Vector Analysis

Dr. A.Afify Ahmed
Mathematics Department
Faculty of Education

Alexandria University

CHAPTER I

Space Curves

Introduction:

In this chapter we shall consider the Elements of Differential Geometry of curves. This will include the basic consideration of the triad of orthonormal vectors associated with any point of a curve and the well known Frenet's formulae giving relations between the are-derivatives of the vectors of the triad and the curvature and torsion.

1. Fundamental triads of lines and planes associated with any point on a curve:

Associated with each point on a curve, there is a set of three mutually perpendicular lines known as:

Tangent, Principal Normal, Binormal

and three mutually perpendicular planes determined by these in a pairs and known as:

Osculating plane, Normal plane, Rectifying plane.

In this section we shall define these important lines and planes and obtain their vector equations.

It may be carefully noted that a curve is analytically representable by an equation of the form

$$\underline{r} = \underline{f}(t)$$

Such that the vector, $\underline{r}$ corresponding to any value of the scalar parameter "t" is the position vector of a point on the curve with respect to some origin 0. The point P on the curve corresponding to the value "t" may be denoted as $P(t)$.

For the following, we shall suppose that $\underline{f}(t)$, possesses continuous derivatives of every order which appear in any investigating.

In general, we shall not be required to consider derivatives of order higher than three.

1-1- Tangent at a point.

Let $P(t)$ be any point on the curve

$$\underline{r} = \underline{f}(t)$$

Take any other point $Q\left(t+\delta t\right)$ on the curve.

The tangent at P is the limiting position of the chord PQ, when $Q \to P$ Let:

$$\underline{r} + \delta \underline{r} = \underline{f}\left(t+\delta t\right)$$

$$\therefore \overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$$

$$= \underline{f}\left(t+\delta t\right) - \underline{f}(t)\delta \underline{r}$$

As, δt, is scalar, the vector:

$$\frac{\delta r}{\delta t} = \frac{\underline{f}(t+\delta t) - \underline{f}(t)}{\delta t},$$

is parallel to $\overrightarrow{PQ}$.
Thus, taking the limit, when, $Q \to P$ and consequently $\delta t \to 0$, we see that the vector:

$\therefore \dfrac{dr}{dt}$ is parallel to the tangent at P.

Hence:

$$\underline{R} = \underline{r} + u\frac{dr}{dt}, \Rightarrow \quad \underline{R} = \underline{f}(t) + u\underline{f}'(t),$$

is the vector equation of the tangent at P.
Here "u" is the scalar parameter; $\underline{r}$, is the position vector of the point P and $\underline{R}$, the position vector of any point on the tangent.

1-2 Osculating plane at a point. Def.

The limiting position of the plane which passes through the tangent at P and is parallel to the tangent at Q, when $Q \to P$, is called the osculating plane at P.

The tangents at $P(t)$ and $Q(t+\delta t)$ are parallel to the vectors: $\underline{f}', \underline{f}'(t+\delta t)$,

so that the plane through the tangent at P and parallel to the tangent at Q is perpend1oular to

the vector:

$$\underline{f}'(t) \wedge \underline{f}'\left(t + \delta t\right) = \underline{f}'(t) \wedge \left[\underline{f}'\left(t + \delta t\right) - \underline{f}'(t)\right],$$

and hence also to the vector:

$$\underline{f}'(t) \wedge \frac{\underline{f}'\left(t + \delta t\right) - \underline{f}'(t)}{\delta t}.$$

Thus, taking the limit, we see that the osculating plane at P is perpendicular to the vector:

$$\underline{f}'(t) \wedge \underline{f}''(t) = \frac{d\overline{r}}{dt} \wedge \frac{d^2\overline{r}}{dt^2}.$$

Hence the equation of the osculating plane at $P(t)$ is:

$$[\underline{R} - \underline{r}] \cdot \left[\underline{f}'(t) \times \underline{f}''(t)\right] = 0,$$

which is the same as:

$$\boxed{(\underline{R} - \underline{r}) \cdot \left(\frac{d\overline{r}}{dt} \wedge \frac{d^2\overline{r}}{dt^2}\right) = 0},$$

$$\therefore \left[\underline{R}\,\frac{d\underline{r}}{dt}\,\frac{d^2\underline{r}}{dt^2}\right] = \left[\underline{r}\,\frac{d\underline{r}}{dt}\,\frac{d^2\underline{r}}{dt^2}\right];$$

$\underline{R}$ being the position vector of any point on the osculating plane.

1-3. Normal plane at a point. Def.

The plane through a point P perpendicular to the tangent thereat is called the Normal plane at P.

Thus the equation of the normal plane at a point P with position vector r, is:

$$\boxed{(\underline{R}-\underline{r})\cdot\left(\frac{d\underline{r}}{dt}\right)=0;}$$

$\underline{R}$, denoting the position vector of any point on the plane. The normal plane is perpendicular to the osculating plane at any point.

1-4. Normal lines at a point. Two special normals. Principal Normal and Binormal.

Any line perpendicular to the tangent line at Q is called a normal at P. The normal plane at a point is the locus of the normal through the point. Of all the normals at a point, two normals, known as principal normal and binormal play a specially important part. We shall now define these.

Principal normal:

The normal lying in the osculating at any point is called the principal normal at the point. Thus the intersection of the normal plane and the osculating plane is the principal normal. Also the plane containing the tangent and the principal normal is the osculating plane.

Binormal:

The normal perpendicular to the principal normal is called Binormal so that the binormal at any point is the line perpendicular to the osculating plane at the point.

The plane containing the tangent and binormal is called Rectifying plane. Thus we have defined tangent, principal normal and binormal at any point; these being mutually perpendicular.

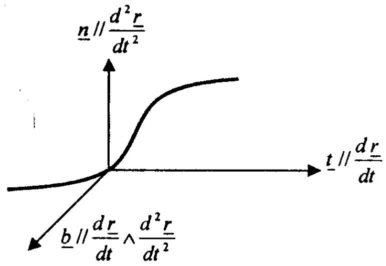
They determine three mutually perpendicular planes, viz.;

i. *Osculating plane* containing the tangent and the principal normal.
ii. *Normal plane* containing the principal normal and binormal.
iii. *Rectifying plane* containing the binormal and the tangent.

1-5. Directions of principal normal and binormal:

From the equation of the osculating plane (1-2), we see that the binormal, being perpendicular to the osculating plane, is parallel to the vector. $\frac{d\underline{r}}{dt} \wedge \frac{d^2\underline{r}}{dt^2}$

Fig.(1)



The principal normal perpendicular to the tangent as well as the binormal and, accordingly, it is parallel to the vector:

$$\left(\frac{d\underline{r}}{dt} \wedge \frac{d^2\underline{r}}{dt^2}\right) \wedge \frac{d\underline{r}}{dt} = \left(\frac{d\underline{r}}{dt}\right)^2 \frac{d^2\underline{r}}{dt^2} - \left(\frac{d\underline{r}}{dt} \cdot \frac{d^2\underline{r}}{dt^2}\right)\frac{d\underline{r}}{dt}.$$

We now take the arc length, s as the parameter. We have:

$$\frac{\delta \underline{r}}{\delta s} = \frac{\overrightarrow{PQ}}{arc\, PQ},$$

So that:

$$\left|\frac{\delta \underline{r}}{\delta s}\right| = \frac{chord\, PQ}{arc\, PQ}.$$

Thus taking limit when $Q \to P$, wee see that:

$$\left|\frac{d\underline{r}}{ds}\right| = 1.$$

So that the vector $\frac{d\underline{r}}{ds}$ is of constant length

unity. Thus:

$$\frac{d\bar{r}}{dt}\cdot\frac{d\bar{r}}{ds}=1 \text{ .... [1]}$$

Differentiating, we get:

$$\frac{d\underline{r}}{ds}\cdot\frac{d^2\underline{r}}{ds^2}=0 \text{ .... [2]}$$

The principal normal is parallel to the vector:

$$\left(\frac{d\underline{r}}{ds}\wedge\frac{d^2\underline{r}}{ds^2}\right)\times\frac{d\underline{r}}{ds}=\left(\frac{d\underline{r}}{ds}\cdot\frac{d\underline{r}}{ds}\right)\frac{d^2\underline{r}}{ds^2}-\left(\frac{d\underline{r}}{ds}\cdot\frac{d^2\underline{r}}{ds^2}\right)\frac{d\underline{r}}{ds}$$

$$=\frac{d^2\underline{r}}{ds^2}$$

Thus we see that the tangent, principal normal and binormal are parallel to the vectors:

$$\frac{d\underline{r}}{ds},\frac{d^2\underline{r}}{ds^2},\frac{d\underline{r}}{ds}\wedge\frac{d^2\underline{r}}{ds^2} \text{ respectively.}$$

1-6. Orthonormal triad of vectors $\underline{t},\underline{n},\underline{b}$

We shall now give precise definitions of unit vector $\underline{t},\underline{n},\underline{b}$ lying along the tangent, principal normal and binormal respectively.

The unit tangent vector "$\underline{t}$":

We have seen that the tangent at P is parallel to the vector $\frac{d\underline{r}}{ds}$. The sense of $\frac{d\underline{r}}{ds}$ is the same as that of the curve along which "s" increases

and, as seen in cor.to , its length is unity. We write:

$$\underline{t} = \frac{d\underline{r}}{ds}$$

So that, "t"is the unit vector along the tangent and in the sense in which the arc,"s", is measured positively.

The vector, $\hat{\underline{t}}$, is called the unit tangent vector.

The unit principal normal"$\underline{n}$":

As $\underline{t} = \dfrac{d\underline{r}}{ds}$, we have:

$$\frac{d\underline{t}}{ds} = \frac{d^2\underline{r}}{ds^2}$$

As seen in the cor. to(1-5), the vector $\dfrac{d^2\underline{r}}{ds^2}$ is parallel to the principal normal. This vector is, however, not necessary of unit length.

The unit vector, $\underline{n}$, called unit principal normal, is defined as a unit vector having the sense of$\dfrac{d\underline{t}}{ds}$. Claarly $\underline{n}$ lies along the principal normal.

The unit binormal "$\underline{b}$":

The vector, $\underline{b}$, is defined by the equation:

$$\underline{b} = \underline{t} \times \underline{n},$$

So that $\underline{b}$, is a unit vector lying along the binormal such that the triad of vectors:

$$\underline{t}, \underline{n}, \underline{b}$$

is right handed.

1-7. Curvature and Torsion.

Def.1 Arc-rate of rotation of tangent is defined as curvature.

Del.2. Arc- rate of rotation of binormal is defined as torsion.

The curvature and torsion are usually denoted by K and τ respectively. Their reciprocals, called radius of curvature and radius of torsion are denoted by ρ and σ respectively. Thus:

$$\rho = \frac{1}{K} \quad , \sigma = \frac{1}{\tau}$$

It will be seen that, ρ is always regarded as positive but σ may be positive or negative so that σ is a signed scalar.

We shall now establish the following results:

$$(i)\left|\frac{d\underline{t}}{ds}\right| = K \qquad ,(ii)\left|\frac{d\underline{b}}{ds}\right| = |\tau|.$$

(i) Let, $\delta\theta$, denote the angle, lying between 0 and π between the tangents at P and Q, i.e between the vectors $\underline{t}$ and $\underline{t} + \delta\underline{t}$.

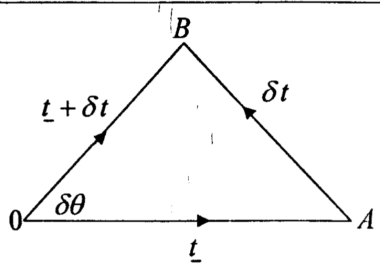


Fig.(2)

Now

$$\underline{t}\times(\underline{t}+\delta\underline{t})=\underline{t}\times\delta\underline{t},$$

represents twice the vector area of the triangle OAB. The magnitude of the area of this triangle being:

$$\frac{1}{2}OA\cdot OB\sin\delta\theta=\frac{1}{2}\sin\delta\theta,$$

we have

$$\left|\underline{t}\times\delta\underline{t}\right|=\sin\delta\theta$$

$$\therefore\left|\underline{t}\times\frac{\delta\underline{t}}{\delta s}\right|=\frac{\sin\delta\theta}{s}=\frac{\sin\delta\theta}{\delta\theta}\frac{\delta\theta}{\delta s}.$$

Taking limit, we obtain

$$\left|\underline{t}\times\frac{d\underline{t}}{ds}\right|=1\cdot k=k.$$

As, t, is of constant length, it is perpendicular to $\frac{d\underline{t}}{ds}$. Hence $\qquad \left|\underline{t}\times\frac{d\underline{t}}{ds}\right|=\left|\frac{d\underline{t}}{ds}\right|.$

Thus: $$\left|\frac{d\underline{t}}{ds}\right| = k.$$

Taking, $\underline{b}$, instead of, $\underline{t}$, we may similarly show that: $$\left|\frac{d\underline{b}}{ds}\right| = |\tau|.$$

2. Frenet's Formulae.

We shall now establish the following important results, known as Frenet's formulae:

$$\frac{d\underline{t}}{ds} = k\underline{n} \quad\text{.......................[1]}$$

$$\frac{d\underline{n}}{ds} = -k\underline{t} - \tau\underline{b} \quad\text{.......................[2]}$$

$$\frac{d\underline{b}}{ds} = \tau\underline{n} \quad\text{.......................[3]}$$

We have seen that $\underline{n}$, is the unit vector having the sense and direction of $\frac{d\underline{t}}{ds}$. Also;

$$\left|\frac{d\underline{t}}{ds}\right| = k.$$

Combining these two facts, we obtain:

$$\frac{d\underline{t}}{ds} = k\underline{n};$$

Now,

$$\underline{t} \cdot \underline{b} = 0.$$

$$\therefore \frac{d\underline{t}}{ds}\cdot \underline{b} + \underline{t}.\frac{d\underline{b}}{ds} = 0$$

or $$k\underline{n}\cdot \underline{b} + \underline{t}\cdot \frac{d\underline{b}}{ds} = 0 \qquad \text{by [1]}$$

or $$\underline{t}\cdot \frac{d\underline{b}}{ds} = 0; \quad \text{for } \underline{n}\cdot \underline{b} = 0.$$

Thus $\frac{d\underline{b}}{ds}$ is perpend1cular to $\underline{t}$. Also, $\underline{b}$, being of constant magnitude, is perpendicular to its derivative $\frac{d\underline{b}}{ds}$,

Hence $\frac{d\underline{b}}{ds}$ is parallel to the vector $\underline{n}$.

Moreover we have seen that:

$$\left|\frac{d\underline{b}}{ds}\right| = |\tau|$$

Thus, we have:

$$\frac{d\underline{b}}{ds} = \tau\underline{n},$$

Which is (3).

Here the torsion τ is positive or negative according as $\underline{t}$ vectors $\underline{n}$ and $\frac{d\underline{b}}{ds}$, as defined above, have the same or opposes

Finally, we have:

$$\underline{n} = \underline{b} \wedge \underline{t}$$

$$\therefore \frac{d\underline{n}}{ds} = \frac{d\underline{b}}{ds} \wedge \underline{t} + \underline{b} \wedge \frac{d\underline{t}}{ds}$$

$$= \tau \underline{n} \wedge \underline{t} + \underline{b} \wedge k\underline{n} = -\tau \underline{b} - k\underline{t} = -k\underline{t} - \tau \underline{b}$$

Which is (2)

3. Expressions for curvature and torsion in terms of the derivatives of , $\underline{r}$, with respect to s:

We have:

$$\frac{d\underline{r}}{ds} = \underline{t} \text{ .... [1]}$$

$$\therefore \frac{d^2\underline{r}}{ds^2} = \frac{d\underline{t}}{ds} = k\underline{n} \text{ .... [2]}$$

$$\therefore k = \left| \frac{d^2\underline{r}}{ds^2} \right| \text{ .... [3]}$$

Multiplying (1) and (2) vector1ally, we obtain:

$$\frac{d\underline{r}}{ds} \wedge \frac{d^2\underline{r}}{ds^2} = k\underline{t} \times \underline{n} = k\underline{b}$$

$$k = \left| \frac{d\underline{r}}{ds} \wedge \frac{d^2\underline{r}}{ds^2} \right| \text{ .... [4]}$$

The equations (3) and (4) give two expressions for k in the required forms.

Differentiating (2) and making use of Frenet's second formula, we obtain:

$$\frac{d^3\underline{r}}{ds^3} = \frac{dk}{ds}\underline{n} + k\frac{d\underline{n}}{ds}$$

$$= \frac{dk}{ds}\underline{n} - k(k\underline{t} + \tau\underline{b})$$

$$\therefore \frac{d^3\underline{r}}{ds^3} = -k^2\underline{t} + \frac{dk}{ds}\underline{n} - k\tau\underline{b} \text{......} [5]$$

$$\therefore \frac{d\underline{r}}{ds} \wedge \frac{d^3\underline{r}}{ds^3} = \frac{dk}{ds}\underline{b} + k\tau\underline{n} \text{.........} [6]$$

Multiplying (2) and (6) scalarly, we have:

$$\frac{d\underline{r}}{ds} \times \frac{d^3\underline{r}}{ds^3} \cdot \frac{d^2\underline{r}}{ds^2} = k^2\tau.$$

Hence,

$$\left[\frac{d\underline{r}}{ds} \cdot \frac{d^2\underline{r}}{ds^2} \wedge \frac{d^3\underline{r}}{ds^3}\right] = -k^2\tau.$$

4. Expressions for ρ and σ in terms of the derivatives of $\underline{r}$ with respect to an arbitrary parameter $\underline{t}$:

We have,

$$\frac{d\underline{r}}{ds} = \frac{d\underline{r}}{dt}\frac{dt}{ds},$$

$$\frac{d^2\underline{r}}{ds^2} = \frac{d^2\underline{r}}{dt^2}\left(\frac{dt}{ds}\right)^2 + \frac{d\underline{r}}{dt}\frac{d^2t}{ds^2}$$

$$\frac{d^3\underline{r}}{ds^3} = \frac{d^3\underline{r}}{dt^3}\left(\frac{dt}{sd}\right)^3 + 3\frac{d^2\underline{r}}{dt^2}\frac{dt}{ds}\frac{d^2t}{ds^2} + \frac{dr}{dt}\frac{d^3t}{ds^3}$$

$$\therefore \frac{d\underline{r}}{ds} \times \frac{d^2\underline{r}}{ds^2} = \frac{d\underline{r}}{dt} \times \frac{d^2\underline{r}}{dt^2}\left(\frac{dt}{ds}\right)^3$$

or

$$k = \left|\frac{d\underline{r}}{ds} \wedge \frac{d^2\underline{r}}{ds^2}\right| = \frac{\left|\frac{d\underline{r}}{dt} \times \frac{d^2\underline{r}}{dt^2}\right|}{\left(\frac{ds}{dt}\right)^3}.$$

Also

$$\frac{d\underline{r}}{ds} \times \frac{d^2\underline{r}}{ds^2} \cdot \frac{d^3\underline{r}}{ds^3} = \frac{d\underline{r}}{dt} \times \frac{d^2\underline{r}}{dt^2} \cdot \frac{d^3\underline{r}}{dt^3}\left(\frac{dt}{ds}\right)^6$$

$$\therefore k^2\tau = -\left[\frac{dr}{ds}\,\frac{d^2r}{ds^2}\,\frac{d^3r}{ds^3}\right] = -\frac{\left[\frac{dr}{dt}\,\frac{d^2r}{dt^2}\,\frac{d^3r}{dt^3}\right]}{\left(\frac{ds}{dt}\right)^6}.$$

Since,

$$\left|\frac{dr}{dt}\right| = \left|\frac{dr}{ds}\frac{ds}{dt}\right| = \left|\frac{dr}{ds}\right|\left|\frac{ds}{dt}\right| = \frac{ds}{dt}, \quad \text{for} \left|\frac{dr}{ds}\right| = 1$$

We can-re-write the results obtained above as follows:

$$k = \left|\frac{d\underline{r}}{dt} \wedge \frac{d^2\underline{r}}{dt^2}\right| \Big/ \left|\frac{d\underline{r}}{dt}\right|^3$$

$$k^2\tau = -\left[\frac{d\underline{r}}{dt}\,\frac{d^2\underline{r}}{dt^2}\,\frac{d^3\underline{r}}{dt^3}\right] \Big/ \left|\frac{d\underline{r}}{dt}\right|^6$$

5. Scalar formulation in terms of Rectangular Cartesian coordinates:

This section is devoted to giving the scalar equivalents, in terms of Rectangular Co-ordinate axes, or the results in the Differential Geometry of curves which we have obtained. As usual $\underline{i}, \underline{j}, \underline{k}$ denote unit vectors along the axes. Let the curve be:

$$\underline{r} = \underline{f}(t) = \underline{i}x + \underline{j}y + \underline{k}z,$$

so that, x, y, z are functions of t. Also we shall denote the derivatives by dashes. Thus we write:

$$\frac{dx}{dt} = x', \frac{d^2x}{dt^2} = x'', \frac{d^3x}{dt^3} = x''', \text{ etc.}$$

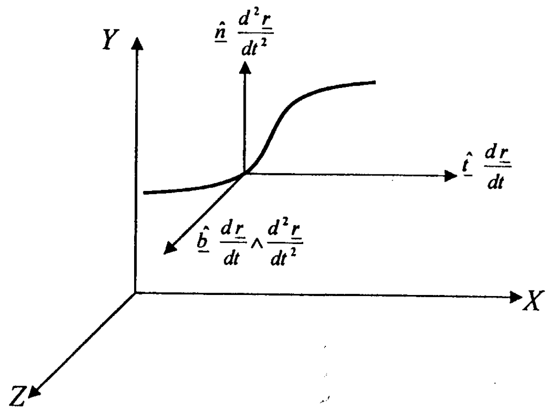


Fig.(3)

Also for the current position vector $\underline{R}$ of any point on the locus associated with the given

curve, we write: $\underline{R} = \underline{i}x + \underline{j}y + \underline{k}z$.

The tangent at $P(x, y, z)$ is:

$$\underline{R} = \underline{r} + u\frac{d\underline{r}}{dt} \quad \text{..........................[1-1]}$$

i.e.

$$\underline{i}X + \underline{j}Y + \underline{k}Z + = (\underline{i}x + \underline{j}y + \underline{k}z) + u(\underline{i}x' + \underline{j}y' + \underline{k}z')$$

or

$$\frac{X - x}{x'} = \frac{Y - y}{y'} = \frac{Z - z}{z'}$$

The osculating plane at $P(x, y, z)$ is:

$$(\underline{R} - \underline{r}) \cdot \frac{d\underline{r}}{dt} \wedge \frac{d^2\underline{r}}{dt^2} = 0 \quad \text{...........[1-2]}$$

i.e.

$$\begin{vmatrix} X - x & Y - y & Z - z \\ x' & y' & z' \\ x'' & y'' & z'' \end{vmatrix} = 0$$

The normal plane at $P(x, y, z)$ is:

$$(\underline{R} - \underline{r}) \cdot \frac{d\underline{r}}{dt} = 0 \quad \text{.......................[1-3]}$$

i.e.

$$\sum (X - x)x' = 0$$

The binormal is parallel to the vector:

$$\frac{d\underline{r}}{dt} \wedge \frac{d^2\underline{r}}{dt^2}, \qquad \text{[1-4]}$$

i.e.

$$\left(\underline{i}x' + \underline{j}y' + \underline{k}z'\right) \wedge \left(\underline{i}x'' + \underline{j}y'' + \underline{k}z''\right)$$

So that the direction ratios of the binormal are:

$$y'z'' - y''z', z'x'' - z''x', x'y'' - x''y'$$

The principal normal is parallel to the vector:

$$\left(\frac{d\underline{r}}{dt}\right)^2 \frac{d^2\underline{r}}{dt^2} - \left(\frac{d\underline{r}}{dt} \cdot \frac{d^2\underline{r}}{dt^2}\right)\frac{d\underline{r}}{dt} \qquad \text{[1-5]}$$

i.e. $$\left(\sum x'^2\right)\sum \underline{i}x'' - \left(\sum x'x''\right)\left(\sum \underline{i}x'\right)$$

So that the direction ratios of the principal normal are:

$$y'\left(x''y' - x'y''\right) + z'\left(x''z' - x'z''\right), \text{ etc.}$$

If, s, be taken as the parameter so that dashes denote derivatives with respect to s, then the direction ratios of t tangent, principal normal and binormal take the form:

$$x', y', z'; x'', y'', z''; y'z'' - y''z',$$
$$z'x'' - z''x', x'y'' - x''y'$$

Frent's formulae:

If $l_1, m_1, n_1; l_2, m_2, n_2; l_3, m_3, n_3$ denotes the direction cosines of the tangent, principal normal and binormal so that

$$\underline{\hat{t}} = \underline{i}l_1 + \underline{j}m_1 + \underline{k}n_1;$$

$$\underline{\hat{n}} = \underline{i}l_2 + \underline{j}m_2 + \underline{k}n_2;$$

$$\underline{\hat{b}} = \underline{i}l_3 + \underline{j}m_3 + \underline{k}n_3$$

Then, making substitutions in the results of ,we obtain:

$$\frac{dl_1}{ds} = \frac{l_2}{\rho}, \frac{dm_1}{ds} = \frac{m_2}{\rho}, \frac{dn_1}{ds} = \frac{n_2}{\rho}$$

$$\frac{dl_2}{ds} = -\frac{l_1}{\rho} - \frac{l_3}{\sigma}, \frac{dm_2}{ds} = -\frac{m_1}{\rho} - \frac{m_2}{\sigma}, \frac{dn_2}{ds}$$

$$= -\frac{n_1}{\rho} - \frac{n_3}{\sigma}$$

$$\frac{dl_3}{ds} = \frac{l_2}{\sigma}, \frac{dm_3}{ds} = \frac{m_2}{\sigma}, \frac{dn_3}{ds} = \frac{n_2}{\sigma}$$

as Frenet's Formulae formulated scalarly.

Curvature. .We have:

$$k = \left|\frac{d\underline{r}}{dt} \wedge \frac{d^2\underline{r}}{dt^2}\right| \Big/ \left|\frac{d\underline{r}}{dt}\right|^3$$

$$= \frac{\left|\sum \underline{i}x' \times \sum \underline{i}x''\right|}{\left|\sum \underline{i}'\right|^3}$$

$$= \frac{\sqrt{\sum (y'z'' - y''z')^2}}{\sum (x'^2)^{3/2}}$$

Torsion: We have:

$$\tau = -\frac{1}{k^2}\frac{\left[\frac{d\underline{r}}{dt}\cdot\frac{d^2\underline{r}}{dt^2}\wedge\frac{d^3\underline{r}}{dt^3}\right]}{\left|\frac{d\underline{r}}{dt}\right|^6}$$

$$= -\frac{1}{k^2}\frac{\begin{vmatrix} x' & y' & z' \\ x'' & y'' & z'' \\ x''' & y''' & z''' \end{vmatrix}}{\left(x'^2+y'^2+z'^2\right)^3}$$

Examples

Ex. [1]:

Find the osculating planet, curvature and torsion at any point of the curve,

$$x = a\cos 2t, y = a\sin 2t, z = 2a\sin t$$

Solution:

For the position vector $\underline{r}$ of any point of the curve,

$$\underline{r} = \underline{i}a\cos 2t + \underline{j}a\sin 2t + \underline{k}2a\sin t$$

We write:

$$\underline{r} = (a\cos 2t, a\sin 2t, 2a\sin t)$$

where the meaning of the notation is self evident.

$$\therefore \frac{d\underline{r}}{dt} = (-2a\sin 2t, 2a\cos 2t, 2a\cos t)$$

$$\frac{d^2\underline{r}}{dt^2} = (-4a\cos 2t, -4a\sin 2t, -2a\cos t)$$

$$\frac{d^3\underline{r}}{dt^3} = (-8a\sin 2t, -8a\cos 2t, -2a\cos t)$$

$$\therefore \frac{d\underline{r}}{dt} \wedge \frac{d^2\underline{r}}{dt^2} = \left[4a^2(\cos t + \cos t\cos 2t),\right.$$

$$\left. -4a^2(\cos t + \cos t\cos 2t), 8a^2\right]$$

Thus osculating plane is:

$$(\underline{R} - \underline{r}) \cdot \frac{d\underline{r}}{dt} \wedge \frac{d^2\underline{r}}{dt^2} = 0$$

i.e.

$$(X - a\cos 2t)(\cos t + \cos 2t\cos t) -$$
$$(Y - a\sin 2t)(\cos t + \cos t\cos 2t) +$$
$$(Z - 2a\sin t)(2) = 0$$

or

$$(\sin t + \sin 2t\cos t)x - (\cos t + \cos t\cos 2t)y$$
$$+ 2z = 3a\sin t$$

Also curvature:

$$k = \left|\frac{d\underline{r}}{dt} \wedge \frac{d^2\underline{r}}{dt^2}\right| \bigg/ \left|\frac{d\underline{r}}{dt}\right|^3$$

$$k = \frac{4a^2\left[(\sin t + \sin 2t\cos t)^2 + (\cos t + \cos 2t\cos t)^2 + 4\right]^{1/2}}{8a^3\left[\sin^2 2t + \cos^2 2t + \cos^2 t\right]^{3/2}}$$

$$k = \frac{1}{2}\sqrt{\frac{5+3\cos^2 t}{\left(1+\cos^2 t\right)^3}}$$

Torsion:

$$\tau = -\frac{\dfrac{d\underline{r}}{dt}\wedge\dfrac{d^2\underline{r}}{dt^2}\cdot\dfrac{d^3\underline{r}}{dt^3}}{\left|\dfrac{d\underline{r}}{dt}\wedge\dfrac{d^2\underline{r}}{dt^2}\right|^2\cdot\left|\dfrac{dr}{dt}\right|^6} = \frac{3}{a(5\sec t + 3\cos t)}$$

Ex.2:

If the tangent to a curve makes a constant angle, α, with a fixed line, then:

$$\sigma = \pm\rho\tan\alpha$$

Conversely, show that if $\dfrac{\sigma}{\rho}$ is constant, the tangent makes a constant angle with a fixed direction.

Solution:

Let, $\underline{e}$, be the unit vector parallel to the given fixed line so that, as given,

$$\hat{\underline{t}}\cdot\hat{\underline{e}} = \cos\alpha \quad \text{..............................[1]}$$

Differentiating, we get:

$$\frac{d\underline{t}}{ds}\cdot\underline{e} = 0,$$

or

$$k\underline{n}\cdot\underline{e}=0$$

Thus

$$\underline{n}\cdot\underline{e}=0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots [2]$$

Hence $\underline{n}$ is perpendicular to $\underline{e}$. Thus the vectors $\underline{b},\underline{t},\underline{e}$ are coplanar.

$$\therefore \underline{b}\cdot\underline{e}=\pm\sin\alpha \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots [3]$$

Differentiating (2) and employing Frenet's second, we get,

$$\frac{d\underline{n}}{ds}\cdot\underline{e}=0$$

i.e.

$$-\left(k\underline{t}+\tau\underline{b}\right)\cdot\underline{e}=0$$

$$\therefore k\cos\alpha\pm\tau\sin\alpha=0 \text{ from (1),(3)}$$

or

$\sigma=\pm\rho\tan\alpha$

Converse. Let $\sigma=a\rho$ where, a, is some constant scalar.

Now,

$$\frac{d\underline{t}}{ds}=\frac{1}{\rho}\underline{n} \quad ,\frac{d\underline{b}}{ds}=\frac{1}{\sigma}\underline{n}$$

$$\therefore \rho\frac{d\underline{t}}{ds}=\underline{n}=\sigma\frac{d\underline{b}}{ds}$$

or

$$\frac{d\underline{t}}{ds} = \frac{\sigma}{\rho}\frac{d\underline{b}}{ds} = a\frac{d\underline{b}}{ds}$$

Integrating, we get

$$\underline{t} = a\underline{b} + \underline{c}$$

where $\underline{c}$ is a constant vector.

Multiplying scalarly with $\underline{t}$, we get

$$\underline{t} \cdot \underline{c} = 1$$

Hence the tangent makes a constant angle with the direction of the fixed vector $\underline{c}$.

Ex. [3]:

The osculating plane at every point of curve touches a fixed sphere; show that the plane though the tangent perpendicular to the principal normal passes though the center of the sphere.

Solution:

Let, a be the center and P the radius of the given sphere.

The osculating plane:

$$(\underline{R} - \underline{r}) \cdot \underline{b} = 0,$$

will touch the sphere if:

$$(\underline{a} - \underline{r}) \cdot \underline{b} = P$$

differentiating w.r to s, we obtain:

$(\underline{a} - \underline{r}) \cdot \frac{d\underline{b}}{ds} - \frac{d\underline{r}}{ds} \cdot \underline{b} = 0$; $\underline{a}, P$ being constants.

$$\therefore (\underline{a}-\underline{r})\cdot \tau\underline{n}-\underline{t}\cdot\underline{b}=0$$

i.e.

$$(\underline{a}-\underline{r})\cdot\underline{n}=0$$

which shows that the plane $(\underline{R}-\underline{r})\cdot\underline{n}=0$ though the tangent and perpendicular to the principal normal passes though the center, $\underline{a}$, of the given sphere.

Ex. [4]:

If x, y, z are the rectangular Cartesian co-ordinates of any point on a curve, show that:

$$x''^2+y''^2+z''^2=\frac{1}{\rho^2\sigma^2}+\frac{1+\rho'^2}{\rho^4};$$

where dashes denote differentiation with respect to arc length.

Solution:

Clearly:

$$x'''^2+y'''^2+z'''^2=\left|\frac{d^3\underline{r}}{ds^3}\right|^2$$

Now

$$\frac{d\underline{r}}{ds}=\underline{t}$$

$$\frac{d^2\underline{r}}{ds^2}=\frac{d\underline{t}}{ds}=\frac{1}{\rho}\underline{n}$$

$$\frac{d^3\underline{r}}{ds^3} = -\frac{\rho'}{\rho^2}\underline{n} + \frac{1}{\rho}\frac{d\underline{n}}{ds}$$

$$= -\frac{\rho'}{\rho^2}\underline{n} - \frac{1}{\rho^2}\underline{t} - \frac{1}{\rho^2\sigma}\underline{b}$$

$$\therefore \left|\frac{d^3\underline{r}}{ds^3}\right|^2 = \frac{\rho'^2}{\rho^4} + \frac{1}{\rho^4} + \frac{1}{\rho^2\sigma^2}$$

$$= \frac{1}{\rho^2\sigma^2} + \frac{1+\rho'^2}{\rho^4}$$

Problems on Chapter I

1. Find the equation of the osculating plane and the direction cosines of the tangent, principal, normal and binormal at any point of the curve

$$x = 2\log t, y = 4t, z = 2t^2 + 1.$$

Find also the curvature and torsion.

2. Find the curvature at the point ,0, of the curve given by:

$$x = \sqrt{3}\cos\theta, y = 2\cos\theta - \sec\theta, z = 2\sin\theta$$

and prove that the principal normal cuts the x-axis at a constant angle

3. Find the curvature of the curve:

$$x = a(u - \sin u), y = a(1 - \cos u), z = bu$$

Also show that the torsion is given by:

$$-b\left[b^2 + a^2(1-\cos u)^2\right]^{-1}.$$

4. Calculate the curvature and torsion for the curve:

$$x = ae^{u}, y = ae^{-u}, z = \sqrt{2}a\,u,$$

and show that the tangent to the curve make a constant angle with a certain fixed direction.

5. Find the curvature k and the torsion τ for the space curve:

$$x = t, y = t^{2}, z = \frac{2}{3}t^{3}.$$

6. For the space curve:

$x = 3\cos t, y = 3\sin t, z = 4t$

Find

   a. The unit tangent $\hat{\underline{t}}$
   b. The principal normal $\hat{\underline{n}}$, curvature k and radius curvature ρ.
   c. The binormal $\hat{\underline{b}}$, torsion τ and radius of torsion.

7. For the space curve in problem 6, find equation in vector and rectangular form for:
   a. Tangent, principal normal and binormal.
   b. Osculating, normal and rectifying plane at the point $t = \pi$.

8. For the space curve:

$$x = x(s), y = y(s), z = z(s)$$

find the radius of curvature ρ and the torsion τ.

9. Show that the radius of curvature of a plane curve with equation $y = f(x)\underline{n}, z = 0$ is given by:

$$\rho = \frac{\left[1+(y')^2\right]^{3/2}}{y''}$$

Show also that $\tau = 0$

Ans. 8:

$$\rho = \left[\left(\frac{d^2x}{ds^2}\right) + \left(\frac{d^2y}{ds^2}\right) + \left(\frac{d^2z}{ds^2}\right)\right]^{-1/2}$$

$$\tau = \frac{1}{\rho^2}\begin{vmatrix} x' & y' & z' \\ x'' & y'' & z'' \\ x'' & y''' & z''' \end{vmatrix}$$

CHAPTER II
CURVILINEAR COORDINATES

1. TRANSFOMATION OF COORDI

Let the rectangular coordinates (x, y, z) any point are expressed as functions of (q_1, q_2, q_3) so that:

$x = x(q_1, q_2, q_3), y = (q_1, q_2, q_3), z = (q_1, q_2, q_3)$ ...[1]

Suppose that (1) can be solved for q_1, q_2, q_3 in terms of x, y, z,

$q_1 = q_1(x, y, z), q_2 = q_2(x, y, z), q_3 = q_3(x, y, z)$ ..[2]

Given a point P with rectangular coordinates (x, y, z) we can from(2) associate a unique set of Coordinates (q_1, q_2, q_3) called the curvilinear coordinates of P. The sets of equations (1) or (2) define a transformation of coordinates.

2. ORTHOGNAL CURVILINEAR COORDINATES:

The surfaces $q_1 = c_1, q_2 = c_2, q_3 = c_3$, wher c_1, c_2, c_3 are constants, are called coordinate surfaces and each pair of these surfaces intersect in curves called coordinate curves or lines (see Fig. 1). If the

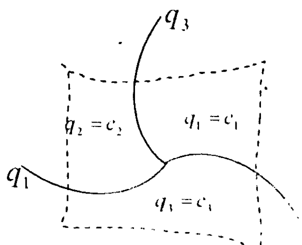


Fig. (1)

coordinate surfaces intersect at right angles the curvilinear coordinate system is called orthogonal. The q_1, q_2 and q_3 coordinate curves of a curvilinear system are analogous to the x, y and z coordinate axes of a rectangular system.

3. UNIT VECTORS IN CURVILINEAR SYSTEMS:

Let

$$\underline{r} = x\underline{i} + y\underline{j} + z\underline{k}$$

be the position vector of a point P. Then (1) can be written

$$\underline{r} = \underline{r}(q_1, q_2, q_3).$$

A tangent vector to the q_1 curve at P (for which q_2 and q_3 are constants) is

$$\frac{\partial r}{\partial q_1}.$$

Then a unit tangent vector in this direction is

$$e_1 = \frac{\partial r}{\partial q_1} \Big/ \left|\frac{\partial r}{\partial q_1}\right|$$

so that $$\frac{\partial r}{\partial q_1} = h_1 \underline{e}_1$$

where $$h_1 = \left|\frac{\partial r}{\partial q_1}\right|.$$

Similarly, if $\underline{e}_2$ and $\underline{e}_3$ are unit tangent vectors to the q_2 and q_3 curves at P respectively, then

$$\frac{\partial r}{\partial q_2} = h_2 \underline{e}_2$$

and $$\frac{\partial r}{\partial q_3} = h_3 \underline{e}_3$$

where $$h_2 = \left|\frac{\partial r}{\partial q_2}\right|$$

and $$h_3 = \left|\frac{\partial r}{\partial q_3}\right|.$$

The quantities h_1, h_2, h_3 are called scale factors.

The unit vectors $\underline{e}_1, \underline{e}_2, \underline{e}_3$ are in the directions of increasing q_1, q_2, q_3, respectively.

Since $\underline{\nabla} q_1$ is a vector at P normal to the surface $q_1 = c_1$, a unit vector in a vector in this direction is given by:

$$\underline{E}_1 = \underline{\nabla} q_1 / |\underline{\nabla} q_1|.$$

Similarly, the unit vectors:

$$\underline{E}_2 = \underline{\nabla} q_2 / |\underline{\nabla} q_2|.$$

and

$$\underline{E}_3 = \underline{\nabla} q_3 / |\underline{\nabla} q_3|.$$

At P are normal to the surfaces $q_2 = c_2$ and $q_3 = c_3$ respectively.

Thus at each point P of a curvilinear system there exist, in general, two sets of unit vectors, $\underline{e}_1, \underline{e}_2, \underline{e}_3$ tangent to the coordinate curves and $\underline{E}_1, \underline{E}_2, \underline{E}_3$ normal to the coordinate surfaces (see Fig.2). The sets become identical if and only if the curvilinear coordinate system is orthogonal. Both sets are analog to the i, j, k unit vectors in rectangular coordinates but are unlike them in that they may change directions from point to point. It can be shown that the sets $\frac{\partial r}{\partial q_1}, \frac{\partial r}{\partial q_2}, \frac{\partial r}{\partial q_3}$ and $\underline{\nabla} q_1, \underline{\nabla} q_2, \underline{\nabla} q_3$ constitute reciprocal systems of vectors.

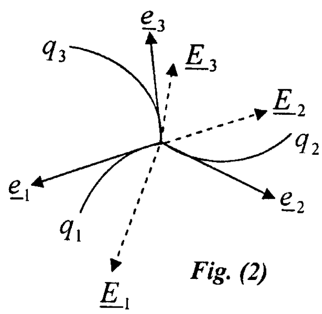


Fig. (2)

A vector $\underline{A}$ can be represented in terms of the unit base vectors $\underline{e}_1, \underline{e}_2, \underline{e}_3$ or $\underline{E}_1, \underline{E}_2, \underline{E}_3$ in the form:

$$\underline{A} = A_1 \underline{e}_1 + A_2 \underline{e}_2 + A_3 \underline{e}_3 = a_1 \underline{E}_1 + a_2 \underline{E}_2 + a_3 \underline{E}_3$$

where A_1, A_2, A_3 and a_1, a_2, a_3 are the respective components of A in each system.

4. ARC LENGH AND VOLUME ELEMENTS:

From $\underline{r} = \underline{r}(q_1, q_2, q_3)$ we have:

$$d\underline{r} = \frac{\partial r}{\partial q_1} dq_1 + \frac{\partial r}{\partial q_2} dq_2 + \frac{\partial r}{\partial q_3} dq_3$$
$$= h_1 dq_1 \underline{e}_1 + h_2 dq_2 \underline{e}_2 + h_3 dq_3 \underline{e}_3$$

$$\frac{\partial r}{\partial q_1} = h_1 \underline{e}_1 \quad \text{...[3]}$$

Then the differential of are length ds is determined from:

$$ds^2 = d\underline{r} \cdot d\underline{r}$$

For orthogonal systems:

$$\underline{e}_1 \cdot \underline{e}_2 = \underline{e}_2 \cdot \underline{e}_3 = \underline{e}_3 \cdot \underline{e}_1 = 0 \quad \text{.......................[4]}$$

and:

$$ds^2 = h_1^2 dq_1^2 + h_2^2 dq_2^2 + h_3^2 dq_3^2 \quad \text{..............[5]}$$

Along a q_1 curve, q_2, and q_3 are constants so that:

$$d\underline{r} = h_1\, dq_1\, \underline{e}_1 .$$

Then the differential of arc

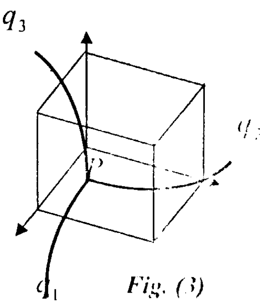


Fig. (3)

length ds_1 along q_1 at P is $h_1 dq_1$.

Similarly the differential of arc length ds_2 along q_2 at P is $h_2 dq_2$, $ds_3 = h_3 dq_3$.

Referring to Fig. (3) the volume element for an orthogonal curvilinear coordinate system is given by:

$$dv = \left| h_1 dq_1 \underline{e}_1 \cdot h_2 dq_2 \underline{e}_2 \wedge h_3 dq_3 \underline{e}_3 \right|$$

$$\therefore dv = h_1 h_2 h_3 dq_1 dq_2 dq_3 \text{ ....[6]}$$

since:

$$\left| \underline{e}_1 \cdot \underline{e}_2 \wedge \underline{e}_3 \right| = 1$$

5.Gradient, Divergence and curl in orthogonal coordinates:

i. Grad ϕ

Let:

$$\underline{\nabla}\phi = f_1 \underline{e}_1 + f_2 \underline{e}_2 + f_2 \underline{e}_2 + f_3 \underline{e}_3$$

where f_1, f_2, f_3 are be determined.

Since:

$$d\underline{r} = \frac{\partial \underline{r}}{\partial q_1} dq_1 + \frac{\partial \underline{r}}{\partial q_2} dq_2 + \frac{\partial \underline{r}}{\partial q_3} dq_3$$
$$= h_1 \underline{e}_1 dq_1 + h_2 \underline{e}_2 dq_2 + h_3 \underline{e}_3 dq_3$$

We have:

$$d\phi = \underline{\nabla}\phi \cdot d\underline{r} = h_1 f_1 dq_1 + h_2 f_2 dq_2 + h_3 f_3 dq_3$$

and:

$$d\phi = \frac{\partial \phi}{\partial q_1} dq_1 + \frac{\partial \phi}{\partial q_2} dq_2 + \frac{\partial \phi}{\partial q_3} dq_3$$

Equating the above two expressions for $d\phi$

$$f_1 = \frac{1}{h_1}\frac{\partial \phi}{\partial q_1}, f_2 = \frac{1}{h_2}\frac{\partial \phi}{\partial q_2}, f_3 = \frac{1}{h_3}\frac{\partial \phi}{\partial q_3}$$

$$\underline{\nabla}\phi = \frac{\underline{e}_1}{h_1}\frac{\partial \phi}{\partial q_1} + \frac{\underline{e}_2}{h_2}\frac{\partial \phi}{\partial q_2} + \frac{\underline{e}_3}{h_3}\frac{\partial \phi}{\partial q_3}$$

ii. Div $\underline{V}$.

From the expression for the gradient ϕ we have:

$$\underline{\nabla} q_1 = \frac{\underline{e}_1}{h_1}, \underline{\nabla} q_2 = \frac{\underline{e}_2}{h_2}, \underline{\nabla} q_3 = \frac{\underline{e}_3}{h_3}$$

Then:

$$\underline{\nabla} q_2 \wedge \underline{\nabla} q_3 = \frac{\underline{e}_2 \wedge \underline{e}_3}{h_2 h_3} = \frac{\underline{e}_1}{h_2 h_3}$$

Therefore:

$$\underline{e}_1 = h_1 \underline{\nabla} q_1 = h_2 n_3 \underline{\nabla} q_2 \wedge \underline{\nabla} q_3$$

Similarly:

$$\underline{e}_2 = h_2 \underline{\nabla} q_2 = h_3 n_1 \underline{\nabla} q_3 \wedge \underline{\nabla} q_1$$

$$\underline{e}_3 = h_3 \underline{\nabla} q_3 = h_1 n_1 \underline{\nabla} q_1 \wedge \underline{\nabla} q_2$$

Let:

$$\underline{V} = V_1 \underline{e}_1 + V_2 \underline{e}_2 + V_3 \underline{e}_3$$

Then:

$$\underline{\nabla} \cdot \underline{V} = \underline{\nabla} \cdot V_1 \underline{e}_1 + \underline{\nabla} \cdot V_2 \underline{e}_2 + \underline{\nabla} \cdot V_3 \underline{e}_3$$

But:

$$\begin{aligned}\underline{\nabla} \cdot \underline{V}_1 \underline{e}_1 &= \underline{\nabla} \cdot \left(V_1 h_2 h_3 \underline{\nabla} q_2 \wedge \underline{\nabla} q_3\right) \\ &= \underline{\nabla}\left(V_1 h_2 h_3\right) \cdot \left(\underline{\nabla} q_2 \wedge \underline{\nabla} q_3\right) \\ &\quad + V_1 h_2 h_3 \underline{\nabla} \cdot \left(\underline{\nabla} q_2 \wedge \underline{\nabla} q_3\right)\end{aligned}$$

since:

$$\begin{aligned}\underline{\nabla} \cdot \left(\underline{\nabla} q_2 \wedge \underline{\nabla} q_3\right) &= \left(\underline{\nabla} \wedge \underline{\nabla} q_2\right) \cdot \underline{\nabla} q_3 \\ &= -\left(\underline{\nabla} \wedge \underline{\nabla} q_3\right) \cdot \underline{\nabla} q_2 \\ &= 0\end{aligned}$$

Therefore:

$$\begin{aligned}\underline{\nabla} \cdot V_1 \underline{e}_1 &= \left(V_1 h_2 h_3\right) \cdot \frac{\underline{e}_1}{h_2 h_3} \\ &= \left[\frac{\underline{e}_1}{h_1} \frac{\partial}{\partial q_1}\left(V_1 h_2 h_3\right) + \frac{\underline{e}_2}{h_2} \frac{\partial}{\partial q_2}\left(V_1 h_2 h_3\right)\right. \\ &\quad \left. + \frac{\underline{e}_3}{h_3} \frac{\partial}{\partial q_3}\left(V_1 h_2 h_3\right)\right] \cdot \frac{\underline{e}_1}{h_2 h_3} \\ &= \frac{1}{h_1 h_2 h_3} \frac{\partial}{\partial q_1}\left(h_2 h_3 V_1\right)\end{aligned}$$

Similarly:

$$\underline{\nabla} \cdot V_2 \underline{e}_2 = \frac{1}{h_1 h_2 h_3} \frac{\partial}{\partial q_2}\left(h_2 h_3 V_2\right)$$

$$\underline{\nabla} \cdot V_3 \underline{e}_3 = \frac{1}{h_1 h_2 h_3} \frac{\partial}{\partial q_3}(h_2 h_3 V_3)$$

Therefore:

$$\underline{\nabla} \cdot \underline{V} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial q_1}(h_2 h_3 V_1) + \frac{\partial}{\partial q_2}(h_3 h_1 V_2) + \frac{\partial}{\partial q_3}(h_1 h_2 V_3)\right]$$

iii. Curl $\underline{V}$.

We have:

$$\underline{\nabla} \wedge \underline{V} = \underline{\nabla} \wedge (V_1 \underline{e}_1 + V_2 \underline{e}_2 + V_3 \underline{e}_3)$$

But:

$$\begin{aligned}
\underline{\nabla} \wedge (V_1 \underline{e}_1) &= \underline{\nabla} \wedge (h_1 V_1 \underline{\nabla} q_1) \\
&= \underline{\nabla}(h_1 V_1) \wedge \underline{\nabla} q_1 + h_1 V_1 \underline{\nabla} q_1 \\
&= \underline{\nabla}(h_1 V_1) \wedge \frac{\underline{e}_1}{h_1} \\
&= \left[\frac{\underline{e}_1}{h_1}\frac{\partial}{\partial q_1}(h_1 V_1) + \frac{\underline{e}_2}{h_2}\frac{\partial}{\partial q_2}(h_1 V_1) \right. \\
&\qquad \left. + \frac{\underline{e}_3}{h_3}\frac{\partial}{\partial q_3}(h_1 V_1)\right] \wedge \frac{\underline{e}_1}{h_1} \\
&= \frac{\underline{e}_2}{h_3 h_1}\frac{\partial}{\partial q_3}(h_1 V_1) - \frac{\underline{e}_3}{h_1 h_2}\frac{\partial}{\partial q_2}(h_1 V_1)
\end{aligned}$$

Similarly:

$$\underline{\nabla} \wedge (V_2 \underline{e}_2) = \frac{\underline{e}_3}{h_2 h_1}\frac{\partial}{\partial q_1}(h_2 V_2) - \frac{\underline{e}_1}{h_3 h_2}\frac{\partial}{\partial q_3}(h_2 V_2)$$

$$\underline{\nabla} \wedge (V_3 \underline{e}_3) = \frac{\underline{e}_1}{h_2 h_3} \frac{\partial}{\partial q_2}(h_3 V_3) - \frac{\underline{e}_2}{h_3 h_1} \frac{\partial}{\partial q_1}(h_3 V_3)$$

Therefore:

$$\begin{aligned}\underline{\nabla} \wedge \underline{V} &= \frac{\underline{e}_1}{h_2 h_3}\left[\frac{\partial}{\partial q_2}(h_3 V_3) - \frac{\partial}{\partial q_3}(h_2 V_2)\right] \\ &+ \frac{\underline{e}_2}{h_3 h_1}\left[\frac{\partial}{\partial q_3}(h_1 V_1) - \frac{\partial}{\partial q_1}(h_3 V_3)\right] \\ &+ \frac{\underline{e}_3}{h_1 h_2}\left[\frac{\partial}{\partial q_1}(h_2 V_2) - \frac{\partial}{\partial q_2}(h_1 V_1)\right] \\ &= \frac{1}{h_1 h_2 h_3}\begin{vmatrix} h_1 \underline{e}_1 & h_2 \underline{e}_2 & h_3 \underline{e}_3 \\ \dfrac{\partial}{\partial q_1} & \dfrac{\partial}{\partial q_2} & \dfrac{\partial}{\partial q_3} \\ h_1 V_1 & h_2 V_2 & h_3 V_3 \end{vmatrix}\end{aligned}$$

6. The expression of Lablace's equation in curvi linear orthogonal coordinate:

$$\begin{aligned}\nabla^2 \phi &= \underline{\nabla} \cdot \underline{\nabla} \phi \\ &= \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial q_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial \phi}{\partial q_1}\right) + \frac{\partial}{\partial q_2}\left(\frac{h_1 h_3}{h_2}\frac{\partial \phi}{\partial q_2}\right)\right. \\ &\qquad \left. + \frac{\partial}{\partial q_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial \phi}{\partial q_3}\right)\right]\end{aligned}$$

7. Special orthogonal coordinate systems:

i. Circular Cylindrical coordinates (ρ, ϕ, z)

Circular Cylindrical coordinates:

$$q_1 = \rho \quad , q_2 = \phi \quad , q_3 = z$$

are defined by relations:

$$x = \rho\cos\phi \ , y = \rho\sin\phi \ , z = z \text{ ....[1]}$$

These may be solved explicitly for the curvilinear coordinates;

$$\rho = \left(x^2 + y^2\right)^{\frac{1}{2}} \ , \phi = \tan^{-1}\left(\frac{y}{x}\right) \ , z = z \text{ ......[2]}$$

$$ds^2 = (d\rho)^2 + \rho^2 (d\phi)^2 + (dz)^2$$

$$\therefore h_2 = h_\rho = 1, h_2 = h_\phi = \rho, h_3 = h_z = 1$$

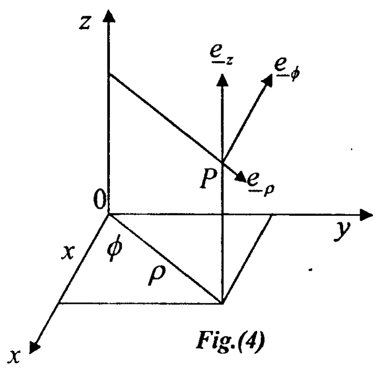


Fig.(4)

Then we can express the quantities (a) $\nabla\Phi$, (b) $\underline{\nabla}\cdot\underline{A}$, (c) $\underline{\nabla}\wedge\underline{A}$, (d) $\nabla^2\Phi$.

Cylindrical coordinates (ρ, ϕ, z)

$$(a) \quad \underline{\nabla}\Phi = \frac{1}{h_1}\frac{\partial \Phi}{\partial q_1}\underline{e}_1 + \frac{1}{h_2}\frac{\partial \Phi}{\partial q_2}\underline{e}_2 + \frac{1}{h_3}\frac{\partial \Phi}{\partial q_3}\underline{e}_3$$

$$= \frac{\partial \Phi}{\partial \rho}\underline{\hat{e}}_\rho + \frac{1}{\rho}\frac{\partial \Phi}{\partial \phi}\underline{\hat{e}}_\phi + \frac{\partial \Phi}{\partial z}\underline{\hat{e}}_z$$

$$(b) \; \underline{\nabla} \cdot \underline{A} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial q_1}(h_2 h_3 A_1) + \frac{\partial}{\partial q_2}(h_3 h_1 A_2) + \frac{\partial}{\partial q_3}(h_1 h_2 A_3)\right]$$

$$= \frac{1}{\rho}\left[\frac{\partial}{\partial \rho}(\rho A_\rho) + \frac{\partial A_\phi}{\partial \phi} + \frac{\partial}{\partial z}(\rho A_z)\right]$$

Where:

$$\underline{A} = A_\rho \underline{e}_1 + A_\phi \underline{e}_2 + A_z \underline{e}_3$$

i.e.

$$A_1 = A_\rho \; , A_2 = A_\phi \; , A_3 = A_z$$

$$(c) \; \underline{\nabla} \wedge \underline{A} = \frac{1}{h_1 h_2 h_3}\begin{vmatrix} h_1 e_1 & h_2 e_2 & h_3 e_3 \\ \dfrac{\partial}{\partial q_1} & \dfrac{\partial}{\partial q_2} & \dfrac{\partial}{\partial q_3} \\ h_1 A_1 & h_2 A_2 & h_3 A_3 \end{vmatrix}$$

$$= \frac{1}{\rho}\begin{vmatrix} e_\rho & \rho e_\phi & e_z \\ \dfrac{\partial}{\partial \rho} & \dfrac{\partial}{\partial \phi} & \dfrac{\partial}{\partial z} \\ A_\rho & \rho A_\phi & A_z \end{vmatrix}$$

$$\underline{\nabla} \wedge \underline{A} = \frac{1}{\rho}\left[\frac{\partial A_z}{\partial \phi} - \frac{\partial}{\partial z}\left(\rho A_\phi\right)\underline{e}_\rho\right.$$

$$\left. + \left(\rho\frac{\partial A_\rho}{\partial z} - \rho\frac{\partial A_z}{\partial \rho}\right)\underline{e}_\phi + \left(\frac{\partial}{\partial \rho}\left(\rho A_\phi\right) - \frac{\partial A_\rho}{\partial \phi}\right)\underline{e}_z\right]$$

$$(d)\ \nabla^2\Phi = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial q_1}\left(\frac{h_2h_3}{h_1}\frac{\partial \Phi}{\partial q_1}\right) + \frac{\partial}{\partial q_2}\left(\frac{h_3h_1}{h_1}\frac{\partial \Phi}{\partial q_2}\right)\right.$$

$$\left. + \frac{\partial}{\partial q_3}\left(\frac{h_1h_2}{h_3}\frac{\partial \Phi}{\partial q_3}\right)\right]$$

$$= \frac{1}{(1)(\rho)(1)}\left[\frac{\partial}{\partial \rho}\left(\frac{(\rho)(1)}{(1)}\frac{\partial \Phi}{\partial \rho}\right) + \frac{\partial}{\partial \phi}\left(\frac{(1)(1)}{(\rho)}\frac{\partial \Phi}{\partial \phi}\right)\right.$$

$$\left. + \frac{\partial}{\partial z}\left(\frac{(1)(\rho)}{(\rho)}\frac{\partial \Phi}{\partial z}\right)\right]$$

$$= \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial \Phi}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 \Phi}{\partial \phi^2} + \frac{\partial^2 \Phi}{\partial z^2}$$

ii. <u>Spherical coordinates</u> (r,θ,ϕ)

Spherical coordinates,

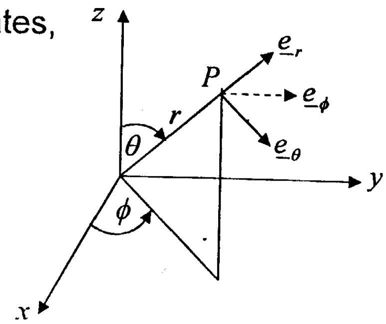


Fig.5

$q_1 = q\ , q_2 = \theta\ , z = r\cos\theta$ ....[1]

This makes:

$x = r\sin\theta\cos\phi, y = r\sin\theta\sin\phi, z = r\cos\theta$ [2]

By restricting the range of these coordinates as follows:

$$0 \le r < \infty \ ,0 \le \theta \le \pi \ ,0 \le \phi \le 2\pi$$

The metrical coefficients:

$$h_1 = 1 \ ,h_\theta = r \ ,h_\phi = r\sin\theta$$

The arc length:

$(ds)^2 = (dr)^2 + r^2(d\theta)^2 + r^2\sin^2\theta(d\phi)^2$ ....[3]

Then we can express gradΨ, div$\cdot\underline{A}$, curl $\underline{A}$ and $\nabla^2\Psi$ in spherical coordinates:

$$\text{(a)}\,\nabla\Psi = \underline{\hat{e}}_r\frac{\partial\Psi}{\partial r} + \frac{\underline{\hat{e}}_\theta}{r}\frac{\partial\Psi}{\partial\theta} + \frac{\underline{\hat{e}}_\phi}{r\sin\theta}\frac{\partial\Psi}{\partial\phi} \text{......[4]}$$

$$\text{(b)}\,\underline{\nabla}\cdot\underline{A} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2A_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta A_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi} \cdots\cdots\cdots\cdots [5]$$

$$\text{(c)}\,\underline{\nabla}\wedge\underline{A} = \frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\underline{e}_1 & h_2\underline{e}_2 & h_3\underline{e}_3 \\ \dfrac{\partial}{\partial q_1} & \dfrac{\partial}{\partial q_2} & \dfrac{\partial}{\partial q_3} \\ h_1A_1 & h_2A_2 & h_3A_3 \end{vmatrix}$$

$$\underline{\nabla} \wedge \underline{A} = \frac{1}{(1)(r)(r\sin\theta)} \begin{vmatrix} \underline{e}_r & r\underline{e}_\theta & r\sin\theta\underline{e}_\phi \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \theta} & \dfrac{\partial}{\partial \phi} \\ A_r & rA_\theta & r\sin\theta A_\phi \end{vmatrix}$$

$$\therefore \underline{\nabla} \wedge \underline{A} = \frac{1}{r^2\sin\theta}\left[\left(\frac{\partial}{\partial\theta}\left(r\sin\theta A_\phi\right) - \frac{\partial}{\partial\phi}\left(rA_\theta\right)\right)\underline{e}_r\right.$$
$$+\left(\frac{\partial A_r}{\partial\phi} - \frac{\partial}{\partial r}\left(r\sin\theta A_\phi\right)\right) r\underline{e}_\theta$$
$$\left.+\left(\frac{\partial}{\partial r}\left(rA_\theta\right) - \frac{\partial A_r}{\partial\theta}\right) r\sin\theta\underline{e}_\phi\right] \cdots [6]$$

$$(\mathrm{d})\, \nabla^2\Psi = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial q_1}\left(\frac{h_2h_3}{h_1}\frac{\partial\Psi}{\partial q_1}\right) + \frac{\partial}{\partial q_2}\left(\frac{h_3h_1}{h_1}\frac{\partial\Psi}{\partial q_2}\right)\right.$$
$$\left.+\frac{\partial}{\partial q_3}\left(\frac{h_1h_2}{h_3}\frac{\partial\Psi}{\partial q_3}\right)\right]$$

$$\nabla^2\Psi = \frac{1}{(1)(r)(r\sin\theta)}\left[\frac{\partial}{\partial r}\left(\frac{(r)(r\sin\theta)}{(1)}\frac{\partial\Psi}{\partial r}\right)\right.$$
$$\left.+\frac{\partial}{\partial\theta}\left(\frac{(r\sin\theta)}{r}\frac{\partial\Psi}{\partial\phi}\right) + \frac{\partial}{\partial\phi}\left(\frac{(1)(r)}{(r\sin\theta)}\frac{\partial\Psi}{\partial z}\right)\right]$$

$$\nabla^2\Psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Psi}{\partial\phi^2} \quad \text{[7]}$$

iii. Parabolic Cylindrical Coordinates

(u, v, z)

$$x = \frac{1}{2}\left(u^2 - v^2\right) \ , y = uv \ , z = z$$

where:

$$-\infty < u < \infty \ , v \geq 0 \ , -\infty < z < \infty$$

$$h_u = h_v = \sqrt{u^2 + v^2} \ , h_z = 1$$

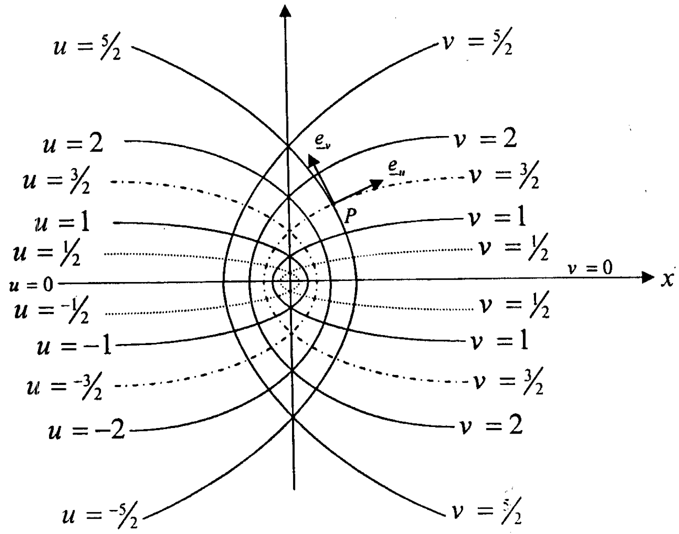


Fig.(6)

In cylindrical coordinates,

$$u = \sqrt{2\rho}\cos\left(\frac{\phi}{2}\right) \quad ,v = \sqrt{2\rho}\sin\left(\frac{\phi}{2}\right) \quad ,z = z$$

The traces of the coordinate surface on xy plane are shown in Fig. 6 below. They are confocal parabolas with a common axis.

iv. Parabolidal Coordinates (u, v, ϕ)

$$x = uv\cos\phi \quad ,y = uv\sin\phi \quad ,z = \frac{1}{2}\left(u^2 - v^2\right)$$

where:

$$u \geq 0 \quad ,v \geq 0 \quad ,0 < \phi < 2\pi$$

$$h_u = h_v = \sqrt{u^2 + v^2} \quad ,h_\phi = uv$$

Two sets of coordinate surfaces are obtained by revolving the parabolas of Fig. 6 above about the x-axis which is relabeled the z-axis. The third sets of coordinate surfaces are plane passing through this axis.

v. Elliptic Cylindrical Coordinates (u, v, z)

$$x = a\cosh u\cos v \quad ,y = a\sinh u\sin v \quad ,z = z$$

where:

$$u \geq 0 \quad ,0 \leq v < 2\pi \quad ,-\infty < z < \infty$$

$$h_u = h_v = a\sqrt{\sinh^2 u + \sin^2 v} \quad ,h_z = 1$$

The traces of the coordinate surfaces on the xy plane are shown in Fig. 7 below. They confocal ellipses and hyperbolas.

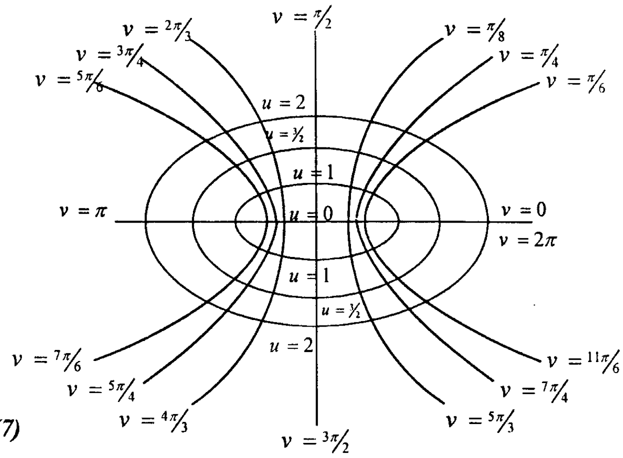


Fig.(7)

Examples:

Ex. [1]

Prove that the spherical coordinates (r,θ,ϕ) constitute an orthogonal system of coordinate. Find the components A_r, A_θ, A_ϕ of the vector $\underline{A} = 2y\underline{i} - z\underline{j} + 3x\underline{k}$ in terms of these coordinates.

Solution:

The transformation formulae between Cartesian coordinates (x,y,z) and spherical coordinates (r,θ,ϕ) are:

$$x = r\sin\theta\cos\phi \ ,y = r\sin\theta\sin\phi \ ,z = r\cos\theta$$

The unit vectors normal to the coordinate. Surface $r = const., \theta = const.$, and $\phi = const.$ are respectively.

$$\underline{e}_r = \frac{1}{h_r}\frac{\partial \underline{r}}{\partial r} = \frac{\partial \underline{r}/\partial r}{\left|\partial \underline{r}/\partial r\right|} \qquad \text{...........[1]}$$

$$= \sin\theta\left(\cos\theta\underline{i} + \sin\phi\underline{j}\right) + \cos\theta\underline{k}$$

$$\underline{e}_r = \frac{1}{h_\theta}\frac{\partial \underline{r}}{\partial \theta} = \frac{\partial \underline{r}/\partial \theta}{\left|\partial \underline{r}/\partial \theta\right|} \qquad \text{..........[2]}$$

$$= \cos\theta\left(\cos\phi\underline{i} + \sin\phi\underline{j}\right) - \sin\theta\underline{k}$$

$$\underline{e}_\phi = \frac{1}{h_\phi}\frac{\partial \underline{r}}{\partial \phi} = \frac{\partial \underline{r}/\partial \phi}{\left|\partial \underline{r}/\partial \phi\right|} \qquad \text{..............................[3]}$$

$$= -\sin\phi\underline{i} + \cos\phi\underline{j}$$

Then:

$$\underline{e}_r \cdot \underline{e}_\theta = 0 \quad , \underline{e}_\theta \cdot \underline{e}_\phi \quad , \underline{e}_r \cdot \underline{e}_\phi = 0$$

Hence $\underline{e}_r, \underline{e}_\theta, \underline{e}_\phi$ are mutually perpendicular and r, θ, ϕ constitute an orthogonal system of coordinates.

Eliminating $\underline{k}$ between (1) ,(2), we get:

$$\sin\theta\underline{e}_r + \cos\theta\underline{e}_\theta = \cos\phi\underline{i} + \sin\phi\underline{j} \qquad \text{............[4]}$$

From equations (3),(4)

$$\underline{i} = \cos\phi(\sin\theta\underline{e}_r + \cos\theta\underline{e}_\theta) - \sin\phi\underline{e}_\phi$$

$$\underline{j} = \sin\phi(\sin\theta\underline{e}_r + \cos\theta\underline{e}_\theta) + \cos\phi\underline{e}_\phi$$

From equations (1),(2)

$$\underline{k} = \cos\theta\underline{e}_r - \sin\theta\underline{e}_\theta$$

Therefore

$$\underline{A} = 2y\underline{i} - z\underline{j} + 3x\underline{k}$$

$$= 2r\sin\theta\sin\phi\left[\cos\phi(\sin\theta\underline{e}_r + \cos\theta\underline{e}_\theta) - \sin\theta\underline{e}_\phi\right]$$

$$-r\cos\theta\left[\sin\phi(\sin\theta\underline{e}_r + \cos\theta\underline{e}_\theta) + \cos\phi\underline{e}_\phi\right]$$

$$+3r\sin\theta\sin\theta\cos\phi(\cos\theta\underline{e}_r - \sin\theta\underline{e}_\theta)$$

$$\therefore A_r = 2r\sin^2\theta\sin\phi\cos\phi - r\cos\theta\sin\theta\sin\phi + 3r\sin\theta\cos\theta\cos\phi$$

$$A_\theta = 2r\sin\theta\cos\phi\sin\phi\cos\phi - r\cos^2\theta\sin\phi - 3r\sin^2\theta\sin\phi\cos\phi$$

$$A_\phi = -2r\sin\theta\sin^2\phi - r\cos\theta\cos\phi$$

Ex. [2]

Prove that:

(i) $\dot{\underline{e}}_r = -\dot{\theta}\underline{e}_\theta + \dot{\phi}\sin\theta\underline{e}_\phi$

$\dot{\underline{e}}_\theta = -\dot{\theta}\underline{e}_r + \dot{\phi}\cos\theta\underline{e}_\phi$

$\dot{\underline{e}}_\phi = -\dot{\phi}(\sin\theta\underline{e}_r + \cos\theta\underline{e}_\phi)$

(ii) $\dot{\underline{e}}_\rho = \dot{\phi}\underline{e}_\phi$, $\dot{\underline{e}}_\phi = -\dot{\phi}\dot{\underline{e}}_\rho$

Where dots denotes differentiation with respect to time t.

Solution:

(i) From example (1) we obtained:

$$\underline{e}_r = \sin\theta\left(\cos\theta\underline{i} + \sin\phi\underline{j}\right) + \cos\theta\underline{k}$$

$$\underline{e}_\theta = \cos\theta\left(\cos\phi\underline{i} + \sin\phi\underline{j}\right) - \sin\theta\underline{k}$$

$$\underline{e}_\phi = -\sin\phi\underline{i} + \cos\phi\underline{j}$$

Hence:

$$\dot{\underline{e}}_r = \left[\cos\theta\left(\cos\phi\underline{i} + \sin\phi\underline{j}\right) - \sin\theta\underline{k}\right]\dot{\theta} + \sin\theta\left(-\sin\phi\underline{i} + \cos\phi\underline{j}\right)\dot{\phi}$$

$$= -\dot{\theta}\underline{e}_\theta + \dot{\phi}\underline{e}_\phi$$

$$\dot{\underline{e}}_\theta = \left[\sin\theta\left(\cos\phi\underline{i} + \sin\phi\underline{j}\right) + \cos\theta\underline{k}\right]\dot{\theta} + \cos\theta\left(-\sin\phi\underline{i} + \cos\phi\underline{j}\right)\dot{\phi}$$

$$= -\dot{\theta}\underline{e}_r + \dot{\phi}\cos\theta\underline{e}_\phi$$

$$\dot{\underline{e}}_\phi = \left(\cos\phi\underline{i} + \sin\phi\underline{j}\right)\dot{\phi} = -\dot{\phi}\sin\theta\underline{e}_r - \dot{\phi}\cos\theta$$

(ii) First we find $\underline{e}_\rho, \underline{e}_\theta, \underline{e}_z$

Where: $(x = \rho\cos\phi, y = \rho\sin\phi, z = z)$

$$\underline{e}_\rho = \frac{\partial\underline{r}/\partial\rho}{\left|\partial\underline{r}/\partial\rho\right|} = \cos\phi\underline{i} + \sin\phi\underline{j} \qquad \text{..................(1)}$$

$$\underline{e}_\phi = \frac{\partial \underline{r}/\partial \phi}{\left|\partial \underline{r}/\partial \phi\right|} = -\sin\phi \underline{i} + \cos\phi \underline{j} \quad \text{..............[2]}$$

$$\underline{e}_\rho = \frac{\partial \underline{r}/\partial z}{\left|\partial \underline{r}/\partial z\right|} = \underline{k} \quad \text{....................................[3]}$$

$$\therefore \quad \dot{\underline{e}}_\rho = -\dot{\phi}\sin\phi \underline{i} + \cos\phi\dot{\phi}\underline{j} = \dot{\phi}\underline{e}_\phi$$

$$\dot{\underline{e}}_\phi = -\cos\phi\dot{\phi}\underline{i} - \sin\phi\dot{\phi}\underline{j} = -\dot{\phi}\underline{e}_\rho$$

Ex. [3]

Express the velocity and acceleration of a practicle in cylindrical coordinate.

Solution:

$$\underline{r} = x\underline{i} + y\underline{j} + z\underline{k} \ , \underline{v} = \frac{d\underline{r}}{dt} \ , \underline{a} = \frac{d^2\underline{r}}{dt^2}$$

from example (2) part (ii), solving equations (1),(2) for $\underline{i}, \underline{j}$ we get:

$$\underline{i} = \cos\phi \, \underline{e}_\rho - \sin\phi \, \underline{e}_\phi$$

$$\underline{j} = \sin\phi \, \underline{e}_\rho + \cos\phi \, \underline{e}_\phi$$

$$\therefore \underline{r} = \rho\cos\phi\left(\cos\phi\underline{e}_\rho - \sin\phi\underline{e}_\phi\right)$$

$$\rho\sin\phi\left(\sin\phi\underline{e}_\phi + \cos\phi\underline{e}_\phi\right) + z\underline{e}_z$$

$$\therefore \underline{r} = \rho\underline{e}_\rho + z\underline{e}_z \quad \text{......................................[1]}$$

Then

$$\underline{v} = \frac{d\underline{r}}{dt} = \dot{\rho}\underline{e}_\rho + \rho\underline{\dot{e}}_\rho + \dot{z}\underline{e}_z$$

$$\therefore \underline{v} = \dot{\rho}\underline{e}_\rho + \rho\dot{\phi}\underline{e}_\phi + \dot{z}\underline{e}_z \text{.............................[2]}$$

And

$$\underline{a} = \frac{d}{dt}\underline{v} = \frac{d}{dt}\left(\dot{\rho}\underline{e}_\rho + \rho\dot{\phi}\underline{e}_\phi + \dot{z}\underline{e}_z\right)$$

$$= \dot{\rho}\underline{\dot{e}}_\rho + \ddot{\rho}\underline{e}_\rho + \rho\dot{\phi}\underline{\dot{e}}_\phi + \rho\ddot{\phi}\underline{e}_\phi + \dot{\rho}\dot{\phi}\underline{e}_\phi + \ddot{z}\underline{e}_z$$

$$= \dot{\rho}\underline{\dot{e}}_\rho\underline{e}_\phi + \ddot{\rho}\underline{e}_\rho + \rho\dot{\phi}\left(-\dot{\phi}\underline{e}_\rho\right) + \rho\ddot{\phi}\underline{e}_\phi + \dot{\rho}\dot{\phi}\underline{e}_\phi + \ddot{z}\underline{e}_z$$

$$\therefore \underline{a} = \left(\ddot{\rho} - \rho\dot{\phi}^2\right)\underline{e}_\rho + \left(\rho\ddot{\phi} - 2\dot{\rho}\dot{\phi}\right)\underline{e}_\phi + \ddot{z}\underline{e}_z \text{ .. [3]}$$

Ex. [4]

The components of a $\underline{V}$ in spherical coordinates (r,θ,ϕ) are given by:

$$V_r = c\left(1 - \frac{a^3}{r^3}\right)\cos\theta, V_\theta = -c\left(1 + \frac{a^3}{2r^3}\right)\sin\theta \text{ }, V_\phi = 0$$

Prove that $\underline{V}$ is irrotational and find its scalar potential. Is $\underline{V}$ solenoidal?

Solution:

The transformation formulae between (r,θ,ϕ) and (x,y,z) are:

$$x = r\cos\theta\cos\phi \text{ }, y = r\sin\theta\sin\phi \text{ }, z = r\cos\theta$$

$$(ds)^2 = (dr\cos\theta\cos\phi + rd\theta\cos\theta\cos\phi - r\sin\theta\sin\phi d\phi)^2$$
$$+(dr\sin\theta\sin\phi + rd\theta\cos\theta\sin\phi + r\sin\theta\cos\phi d\phi)^2$$
$$+(dr\cos\theta - r\sin\theta d\theta)^2$$
$$=(dr)^2 + r^2(d\theta)^2 + r^2\sin^2\theta(d\phi)^2$$

Therefore the metrical coefficients are:

$$h_r = 1 \ ,h_\theta = r \ ,h_\phi = r\sin\theta$$

$$\nabla\wedge\underline{V} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \underline{e}_r & r\underline{e}_\theta & r\sin\theta\underline{e}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ V_r & rV_\theta & r\sin\theta V_\phi \end{vmatrix}$$

$$= \frac{1}{r^2\sin\theta}\begin{vmatrix} \underline{e}_r & r\underline{e}_\theta & r\sin\theta\underline{e}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ c\left(1-\frac{a^3}{r^3}\right)\cos\theta & -cr\left(1+\frac{a^3}{2r^2}\right)\sin\theta & 0 \end{vmatrix}$$

$$\nabla\wedge\underline{V} = \frac{cr\sin\theta\underline{e}_\phi}{r^2\sin\theta}\left[-\frac{\partial}{\partial r}\left(r+\frac{a^3}{2r^2}\right)\sin\theta - \frac{\partial}{\partial\theta}\left(1-\frac{a^3}{r^3}\right)\cos\theta\right]$$
$$= \frac{c}{r}\underline{e}_\phi\left[\left(1-\frac{a^3}{r^3}\right)\sin\theta - \left(1-\frac{a^3}{r^3}\right)(-\sin\theta)\right] = 0$$

i.e. the vector $\underline{V}$ is irrotatational, its scalar potential ϕ is given by

$$\phi = \int \underline{V} \cdot d\underline{r} = \int V_r dr + V_\theta d\theta + V_\phi d\phi$$

$$= \int \left(1 - \frac{a^3}{r^3}\right) \cos\theta dr - \left(1 + \frac{a^3}{2r^2}\right) \sin\theta d\theta$$

$$= \int d\left(r + \frac{a^3}{2r^2}\right) \cos\theta$$

$$= c\left(r + \frac{a^3}{2r^2}\right) \cos\theta$$

$$\underline{\nabla} \cdot \underline{V} = \frac{1}{r^2 \sin\theta}\left[\frac{\partial}{\partial r}\left(r^2 \sin\theta V_r\right) + \frac{\partial}{\partial \theta}\left(r \sin\theta V_\theta\right) + \frac{\partial}{\partial \phi}\left(r V_\phi\right)\right]$$

$$= \frac{c}{r^2 \sin\theta}\left[\frac{\partial}{\partial r}\left(r^2\left(1 - \frac{a^3}{r^3}\right)\sin\theta\cos\theta\right) - \left(r + \frac{a^3}{2r^2}\right) 2\sin\theta\cos\theta\right]$$

$$= 0$$

i.e. The vector $\underline{V}$ is solenoidal.

Ex. [5]

Express $\underline{\nabla}\phi, \nabla^2\phi$ in the parabolic cylindrical coordinates (u, v, z). Hence

(i) Find directional derivative of $\phi = \cos u e^{-(v+z)}$ at $(0,1,-1)$ in the direction of $(\underline{e}_u + 2\underline{e}_v + 2\underline{e}_z)$.

(ii) Find the most general function $f(u)$ such that $\phi = f(u)e^{-v}$ is harmonic.

Solution:

$$r=\frac{1}{2}\left(u^2-v^2\right)\underline{i}+uv\underline{j}+z\underline{k}$$

$$x=\frac{1}{2}\left(u^2-v^2\right)\ ,y=uv\ ,z=z$$

$$(ds)^2=(dx)^2+(dy)^2+(dz)^2$$

$$=(udu-vdv)^2+(udv+vdu)^2+(dz)^2$$

$$=\left(u^2+v^2\right)(du)^2+\left(u^2+v^2\right)(dv)^2+(dz)^2$$

$$=\left(u^2+v^2\right)\left[(du)^2+(dv)^2\right]+(dz)^2$$

$$\therefore h_u=h_v=\sqrt{u^2+v^2}\ ,h_z=1$$

$$\underline{\nabla}\phi=\frac{\underline{e}_u}{\sqrt{u^2+v^2}}\frac{\partial\phi}{\partial u}+\frac{\underline{e}_v}{\sqrt{u^2+v^2}}\frac{\partial\phi}{\partial v}+\underline{e}_z\frac{\partial\phi}{\partial z}$$

$$\nabla^2\phi=\frac{1}{u^2+v^2}\left[\frac{\partial}{\partial u}\left(\frac{\partial\phi}{\partial u}+\frac{\partial}{\partial v}\frac{\partial\phi}{\partial v}\right)+\frac{\partial}{\partial z}\left(u^2+v^2\right)\frac{\partial\phi}{\partial z}\right]$$

$$=\frac{1}{u^2+v^2}\left(\frac{\partial^2\phi}{\partial u^2}+\frac{\partial^2\phi}{\partial v^2}\right)+\frac{\partial^2\phi}{\partial z^2}$$

$$\frac{\partial\phi}{\partial s}=\underline{s}\cdot\nabla\phi=\frac{1}{3}(\underline{e}_u+2\underline{e}_v+2\underline{e}_z)\cdot\left\{\frac{1}{\sqrt{u^2+v^2}}\left[\underline{e}_u\frac{\partial\phi}{\partial u}+\underline{e}_v\frac{\partial\phi}{\partial v}\right]+\frac{\partial\phi}{\partial z}\underline{e}_z\right\}$$

$$\frac{\partial\phi}{\partial s}=\frac{1}{3\sqrt{u^2+v^2}}\left(\frac{\partial\phi}{\partial u}+2\frac{\partial\phi}{\partial v}\right)+\frac{2}{3}\frac{\partial\phi}{\partial z}$$

But

$$\frac{\partial \phi}{\partial u} = -\sin u\, e^{-(v+z)},$$

$$\frac{\partial \phi}{\partial v} = -\cos u\, e^{-(v+z)},$$

$$\frac{\partial \phi}{\partial z} = -\cos u\, e^{-(v+z)}$$

Then

$$\frac{\partial \phi}{\partial s} = \frac{(-\sin u - 2\cos u)\, e^{-(v+z)}}{3\sqrt{u^2 + v^2}} - \frac{2}{3}\cos u e^{-(v+z)}$$

Therefore

$$\left(\frac{\partial \phi}{\partial s}\right)_{(0,1,-1)} \doteq -\frac{4}{3}$$

$$\nabla^2 \phi = 0 \Rightarrow \frac{1}{u^2 + v^2}\left(\frac{\partial^2 \phi}{\partial u^2} + \frac{\partial^2 \phi}{\partial v^2}\right) + \frac{\partial^2 \phi}{\partial z^2} = 0$$

$$\therefore \frac{1}{u^2 + v^2}\left[\frac{\partial^2}{\partial u^2}(f(u))e^{-v} + \frac{\partial^2}{\partial v^2}(f(u))e^{-v}\right] + 0$$
$$= 0$$

$$\therefore \frac{d^2 f}{du^2} + f(u) = 0$$

Therefore the solution is

$$f(u) = A\cos u + B\sin u$$

$$\therefore \phi = (A\cos u + B\sin u)e^{-v}$$

Ex. [6]

Parabolidal coordinates (u,v,ϕ) defined by

$$x = uv\cos\phi \;, y = uv\sin\phi \;, z = \frac{1}{2}\left(u^2 - v^2\right)$$

Prove that (u,v,ϕ) is an orthogonal curvilinear coordinate then obtain expression for $\nabla^2\Psi$ in these coordinates where Ψ is a scalar field. If Ψ is a function of u only, show that the solution of the Laplace's sequation $\nabla^2\Psi = 0$ under the condition $\Psi = 0$ in $u = 1$ is $\Psi = A\log u$ where A is constant.

Solution:

$$\underline{e}_u = \frac{1}{h_u}\frac{\partial \underline{r}}{\partial u} = \frac{\partial \underline{r}/\partial u}{\left|\partial \underline{r}/\partial u\right|}$$

$$= \frac{1}{\sqrt{u^2+v^2}}\left(v\cos\phi\,\underline{i} + v\sin\phi\,\underline{j} + u\underline{k}\right)$$

$$\underline{e}_v = \frac{1}{h_v}\frac{\partial \underline{r}}{\partial v} = \frac{\partial \underline{r}/\partial v}{\left|\partial \underline{r}/\partial v\right|}$$

$$= \frac{1}{\sqrt{u^2+v^2}}\left(u\cos\phi\,\underline{i} + u\sin\phi\,\underline{j} - v\underline{k}\right)$$

$$\underline{e}_\phi = \frac{1}{h_\phi}\frac{\partial \underline{r}}{\partial \phi} = \frac{\partial \underline{r}/\partial \phi}{\left|\partial \underline{r}/\partial \phi\right|}$$

$$= \frac{1}{uv}\left(-uv\sin\phi\,\underline{i} + uv\cos\phi\,\underline{j}\right)$$

Then

$$\underline{e}_u \cdot \underline{e}_v = \underline{e}_v \cdot \underline{e}_\phi = \underline{e}_\phi \cdot \underline{e}_u = 0$$

i.e. the system is orthogonal,

and

$$h_u = h_v = \sqrt{u^2 + v^2} \ , h_\phi = uv$$

$$\nabla^2\Psi = \frac{1}{uv\left(u^2+v^2\right)}\left[\frac{\partial}{\partial u}\left(uv\frac{\partial \Psi}{\partial u}\right) + \frac{\partial}{\partial v}\left(uv\frac{\partial \Psi}{\partial v}\right) + \frac{\partial}{\partial \phi}\left(\frac{u^2+v^2}{uv}\frac{\partial \Psi}{\partial \phi}\right)\right]$$

$$\therefore \nabla^2\Psi = \frac{1}{u\left(u^2+v^2\right)} + \frac{\partial}{\partial u}\left(u\frac{\partial \Psi}{\partial u}\right) + \frac{1}{v\left(u^2+v^2\right)}\frac{\partial}{\partial v}\left(v\frac{\partial \Psi}{\partial v}\right) + \frac{1}{u^2v^2}\frac{\partial^2 \Psi}{\partial \phi^2}$$

Now if Ψ is a function of u only then

$$\nabla^2\Psi = \frac{1}{u\left(u^2+v^2\right)}\frac{\partial}{\partial u}\left(u\frac{\partial \Psi}{\partial u}\right)$$

Therefore

$$\frac{d}{du}\left(u\frac{\partial \Psi}{\partial u}\right) = 0,$$

and the general solution is

$$\Psi = A\log u + B$$

where A, B are arbitrary constants.

But $\Psi = 0$ on $u = 1$ gives $0 = A\log 1 + B$

therefore $B = 0$ and,

$$\Psi = A \log \boldsymbol{u}$$

Ex. [7]

Find the potential of a vector specified in cylindrical coordinates:

$$\vec{A} = \left(\frac{1}{\rho} \tan^{-1} z + \cos\phi \right) \vec{e_\rho} - \sin\phi \vec{e_\phi} + \frac{\ln\rho}{1+z^2} \vec{e_z}$$

Solution:

$$\nabla \times \vec{A} = \frac{1}{\rho} \begin{vmatrix} \underline{e}_\rho & \rho \underline{e}_\phi & \underline{e}_z \\ \dfrac{\partial}{\partial \rho} & \dfrac{\partial}{\partial \phi} & \dfrac{\partial}{\partial z} \\ A_\rho & \rho A_\phi & A_z \end{vmatrix} = 0$$

Then $\vec{A} = \nabla V$

$$\therefore \frac{\partial V}{\partial \rho} = A_\rho$$

$$\frac{\partial V}{\partial \phi} = -\rho \sin\phi = \rho A_\phi$$

$$\frac{\partial V}{\partial z} = A_z$$

$$\therefore V = \ln\rho \tan^{-1} z + \rho\cos\phi + \boldsymbol{c}$$

Problems

1. Prove that the cylindrical coordinates (ρ, ϕ, z) constitute an orthogonal system of coordinates. Find the components A_ρ, A_ϕ, A_z of the vector $\underline{A} = z\underline{i} - 2x\underline{j} + y\underline{k}$ in terms of these coordinates.
2. Prove that:
   a. Parabolic cylindrical,
   b. Elliptic cylindrical,
   c. Oblate spherical coordinates system are orthogonal.
3. Find:
   a. The metrical coefficients and,
   b. The volume element for prolate spherical coordinates.
4. find the element of a volume element in
   a. cylindrical,
   b. spherical, and
   c. paraboloided coordinates.
5. Given the coordinate transformation $q_1 = xy, q_2 = \frac{1}{2}(x^2 + y^2), q_3 = z$, show that the coordinate system is not orthogonal.
6. Find $\nabla^2\Psi$ in oblate spherical coordinates.

7. Write $\dfrac{\partial^2\phi}{\partial x^2}+\dfrac{\partial^2\phi}{\partial y^2}=0$ in elliptic cylindrical coordinates.
8. Given that $f(r,\theta,\phi)=ar^2\sin\theta\cos^2\phi$ where (r,θ,ϕ) are spherical coordinates, calculate the derivative of f in the direction $\underline{s}=\dfrac{1}{4}\left(3\underline{e}_r+\sqrt{3}\underline{e}_\theta+2\underline{e}_\phi\right)$ at the point $\left(a,\dfrac{\pi}{3},\dfrac{\pi}{4}\right)$.
9. Express Maxwell's equation $\nabla\wedge\underline{E}=-\dfrac{1}{c}\dfrac{\partial\underline{u}}{\partial t}$ in prolate spherical coordinates.
10. Express schroedinger's equation of quantum Mechanics $\nabla^2\Psi+\dfrac{8\pi^2 m}{h^2}\left(E-V(x,y,z)\right)\Psi=0$ in prolate spherical coordinates, where m,h and E are constants.
11. Express Laplace's equation in spherical polar coordinates. Determine the most general function $f(r)$ such that $V=f(r)\cos\theta$ satisfies this equation.
12. Express the heat equation

$\frac{\partial U}{\partial t} = K\nabla^2 U$ in spherical coordinates if U independent of:

a. ϕ,

b. ϕ and θ,

c. r and t,

d. ϕ and θ and t.

13. Paraboloidal coordinates (u, v, ϕ) defined by

$$x = uv\cos\phi, y = uv\sin\phi, z = \frac{1}{2}(u^2 - v^2)$$

where $u \geq 0, v \geq 0, 0 \leq \phi < 2\pi$. Express $\nabla\Psi, \nabla^2\Psi$ in these coordinates. Hence.

a. Find the direction derivative of $\Psi = ue^v \sin\phi$ in the direction $\underline{n}$ of $(\underline{e}_u - 2\underline{e}_v + 2\underline{e}_\phi)$ at the point $\left(0, 1, \frac{\pi}{2}\right)$.

b. Find the solution of $\nabla^2\Psi = 0$ regular at $u = 0, v = 0$ in the form $\Psi = f(u) + g(v)$.

14. prove that the vector $\underline{E}$ whose components in terms of the cylindrical coordinates (ρ, ϕ, z) are

$$E_\rho = \left(1 - \frac{a^2}{\rho^2}\right)\cos\phi, E_\phi = -\left(1 + \frac{a^2}{\rho^2}\right)\sin\phi, E_z$$

where a, is a constant, is irrotational and its scalar potential. Is $\underline{E}$ solenoidal?.

CHAPTER III
Tensor analysis

1. Introduction:

The concept of a tensor has its origin in the developments of differentia1 geometry by Gauss, Riemann and Christoffel.

The 1nvestigatior of relations with remain valid when we change from one coordinate system to any other is the ch1ef aim of Tensor analysis. Physical laws must be independent of any particular coordinate systems used in describing them mathematica1 1y, if they are to be valid.

A study of the consequences of this requirement leads to utilize the tensor analysis as the mathematical background in which such laws can be formulated. In particular, Einstein found it an excellent tool for the presentation of his General Relativity theory.

2. Space of N Dimensions:

A point in N dimensional space is a set of N numbers denoted by $\left(x^1, x^2, \cdots x^N\right)$, these real variables are called the coordinate of the point. The suffixes $1, 2, \cdots, i, \cdots, N$ are called superscripts and merely serve as labels and don't possess any significance as power indices. Later we shall introduce quantities of

the type a_i again the i, are called a subscript will act only as label.

Then all the points corresponding to all values of the coordinates are said to form an N-dimensional space.

The fact that we cannot visualize points in space of dimension higher than three has of course nothing whatsoever to do with the existence.

3. Coordinate Transformations:

Let $\left(x^1, x^2, \cdots, x^N\right)$ and $\left(x^{-1}, x^{-2}, \cdots, x^{-N}\right)$ be coordinates of a point in two different frames of reference.

Suppose there exist N independent relations between the coordinates of the two systems having the form

$$\begin{aligned} x^{-1} &= x^{-1}\left(x^1, x^2, \cdots x^N\right) \\ x^{-2} &= x^{-2}\left(x^1, x^2, \cdots x^N\right) \\ &\vdots \\ x^{-N} &= x^{-N}\left(x^1, x^2, \cdots x^N\right) \end{aligned} \qquad \text{.................[1]}$$

Which we can indicate briefly by

$$x^{-k} = x^k\left(x^1, x^2, \cdots x^N\right) \quad ;k = 1, 2, \cdots N \text{ [2]}$$

where it is supposed that the functions involved are single valued, continuous, and have continues derivatives. Then conversely to each set of coordinates $\left(x^{-1}, x^{-2}, \cdots, x^{-N}\right)$ there will correspond a unique set $\left(x^{1}, x^{2}, \cdots, x^{N}\right)$ given by

$x^{k} = x^{k}\left(x^{-1}, x^{-2}, \cdots x^{-N}\right) \quad ; k = 1, 2, \cdots N$ ..[3]

The relations (2) or (3) define a transformation of coordinates from one frame of reference to another.

4. <u>The Summation Convention:</u>

In writing an expression such as

$$a_1 x^1 + a_2 x^2 + \cdots + a_N x^N$$

we can use the short notation

$$\sum_{j=1}^{N} a_j x^j$$

An even shorter notation is simply to write it as $a_j x^j$, where we dopt the convention that wherever an index (subscript or superscript) is repeated in a given term we have to sum over that index from 1 to N unless otherwise specified. This is called the summation convention. Clearly, instead of using the index j we could have used another letter, say P, and the sum could be written $a_P x^P$. Any index

which is repeated in a given term, so that the summation convention applies, is called s dummy index or umbral index.

An index occurring only once in a given term is called free index and can stand for any the numbers $1,2,\cdots,N$ such as k in equation (2) or (3), each of which represents N equations.

5. Contra-variant and Covariant Vectors:

If N quantities $A^1,A^2,\cdots,A^N$ in coordinate system $\left(x^1,x^2,\cdots,x^N\right)$ are related to N other quantities $\overline{A}^1,\overline{A}^2,\cdots\overline{A}^N$ in another coordinate system $\overline{x}^1,\overline{x}^2,\cdots,\overline{x}^N$ by the transformation equations

$$\overline{A}^P = \sum_{q=1}^{N} \frac{\partial \overline{x}^P}{\partial x^q} A^q \qquad ; P = 1,2,\cdots,N$$

which by the conventions adopted can simply be written as

$$\overline{A}^P = \frac{\partial \overline{x}^P}{\partial x^q} A^q$$

they are called components of contra-variant vector or contra-variant tensor of the first rank or first order.

If N quantities $A_1,A_2,\cdots,A_N$ in coordinate system $\left(x^1,x^2,\cdots,x^N\right)$ are related to N other

quantities $\overline{A}^1, \overline{A}^2, \cdots \overline{A}^N$ in another coordinate system $\overline{x}^1, \overline{x}^2, \cdots, \overline{x}^N$ by the transformation equations

$$\overline{A}_p = \sum_{q=1}^{N} \frac{x^q}{\overline{x}^p} A^q \qquad ; p = 1,2,\cdots,N$$

or

$$\overline{A}_p = \frac{\partial x^q}{\partial \overline{x}^p} A^q$$

they are called components of a covariant vector or covariant tensor of the first rank or first order.

Note that a superscript is used to indicate contra-variant components whereas a subscript is used to indicate covariant components; an exception occurs in the notation for coordinates.

Instead of speaking of tensor whose components a A^P or A_p we shall often refer simply to the tensor A^P or A_p. No confusion should arise from this.

6. Contra-variant, Covariant and Mixed Tensor:

If N^2 quantities A^{qs} in coordinate system $(x^1, x^2, \cdots, x^N)$ are related to N^2 other quantities A^{-pr} in another coordinate system

$\left(x^{-1}, x^{-2}, \cdots, x^{-N}\right)$ by the transformation equations.

$$A^{-pr} = \sum_{s=1}^{N} \sum_{r=1}^{N} \frac{\partial \overline{x}^p}{\partial x^q} \frac{\partial \overline{x}^r}{\partial x^s} A^{qs}, p, r = 1, 2, \cdots, N$$

or

$$A^{-pr} = \frac{\partial \overline{x}^p}{\partial x^q} \frac{\partial \overline{x}^r}{\partial x^s} A^{qs}$$

by the adopted conventions, they called contra-variant components of tensor of the second rank or rank two.

The N^2 quantities A_{qs} are called covariant componet of tensor of second rank if

$$\overline{A}_{pr} = \frac{\partial x^q}{\partial \overline{x}^p} \frac{\partial x^s}{\partial \overline{x}^r} A_{qs}$$

Similarly the N^2 quantities A_s^q are called components of mixed tensor of second rank if

$$\overline{A}_r^P = \frac{\partial \overline{x}^p}{\partial x^q} \frac{\partial x^s}{\partial \overline{x}^r} A_s^q$$

7. <u>The kronecker delta</u> δ_k^j:

It is defined by

$$\delta_k^j = \begin{cases} 0 & \text{if } j \neq k \\ 1 & \text{if } j = k \end{cases}$$

An obvious property of the kronecker delta is that

$$\delta_k^j A^k = A^j \quad \text{since}$$

$$\delta_k^j = \delta_1^j A^1 + \delta_2^j A^2 + \cdots = A^j$$

Also

$$\frac{\partial x^k}{\partial x^i} = \delta_i^k$$

because the coordinate x^i are independent.

Ex. Show that

$$\delta_j^i \delta_k^j = \delta_k^i$$

$$\delta_i^i = N$$

$$\frac{\partial x^j}{\partial \overline{x}^i}\frac{\partial \overline{x}^i}{\partial x^k} = \delta_k^j$$

As the notation indicates, the kronecker delta is mixed tensor of second rank.

8. Tensors of Ranks greater than two:

It is easily defined. For example, $A_k^q l^{st}$ are the components of mixed tensor of rank 5, contra-variant of order 3 and covariant of order 2, if they transform according to the relation

$$\overline{A}_i^p j^{rm} = \frac{\partial \overline{x}^p}{\partial x^q}\frac{\partial \overline{x}^r}{\partial x^s}\frac{\partial \overline{x}^m}{\partial x^t}\frac{\partial x^k}{\partial \overline{x}^i}\frac{\partial x^l}{\partial \overline{x}^j} A_k^q l^{st}$$

9. Scalers or Invariants:

Suppose ϕ is a function of coordinates x^k,

and let $\overline{\phi}$ denote the functional value under a transformation to a new set of coordinates $\overline{x}^k$. Then ϕ is called a scaler or invariant with respect to the coordinate transformation if $\phi = \overline{\phi}$. A scaler or invariant is also called a tensor of rank zero.

10. Tensor fields:

If to each point of a region in N dimensions1 space there corresponds a definite tensor, we say that a tensor field has
been defined. This is a vector field or a scaler field according as the tensor is of rank one or zero. It should be noted that a tensor or a tensor field is not just the set of its components in one special coordinate system but all the poss1ble sets under any transformation of coordinates.

11. Symmetric and Skew-Symmetric Tensors

A tensor is called symmetric with respect to two contra-variant or two covariant indices if its components remain unaltered upon interchange of the indices. Thus if

$$A_{q\ s}^{mpr} = A_{q\ s}^{pmr}$$

the tensor is symmetric in m and p.

If a tensor is symmetric with respect to any two contra-variant and any two covariant indices, it is called symmetric.

A tensor is celled skew- symmetric with respect to contra- variant or two covariant indices if its components change sign upon interchange of the indices. Thus 1f

$$A^{m\,pr}_{q\,s} = -A^{pmr}_{q\,s}$$

the tensor is skew-symmetric in m and p

If a tensor is skew-symmetric with respect to any two contra-variant and any covariant indices it is called a skew-symmetric.

Ex. A tensor of the 2^{nd} order A_{ij} can then be splitted into one symmetric and another skew-symmetric as:

$$A_{ij} = \frac{1}{2}\left(A_{ij} + A_{ji}\right) + \frac{1}{2}\left(A_{ij} - A_{ji}\right)$$

12. Examples

Example (1):

Write the law of transformation for tensor (a) A^{i}_{jk}, (b) A^{mn}_{ijk}, (c) C^{n}.

(a) $\overline{A}^{p}_{ar} = \dfrac{\partial \overline{x}^{p}}{\partial x^{m}} \dfrac{\partial x^{j}}{\partial \overline{x}^{p}} \dfrac{\partial x^{k}}{\partial \overline{x}^{r}} A^{i}_{jk}$

(b) $\overline{B}^{pq}_{rst} = \frac{\partial \overline{x}^p}{\partial x^m} \cdot \frac{\partial \overline{x}^p}{\partial x^n} \cdot \frac{x^i}{\partial \overline{x}^r} \frac{x^j}{\partial \overline{x}^s} \frac{x^k}{\partial \overline{x}^t} B^{mn}_{ijk}$

(c) $\overline{C}^p = \frac{\partial \overline{x}^p}{\partial x^m} c^n$

Example (2):

A covariant vector has components $xy, 2y - z^2, xz$ in rectangular coordinates. Find its covariant components in spherical coordinates.

Let A_j denotes the covariant components in rectangular coordinates.

$$x^1 = x, x^2 = y, x^3 = z$$

Then

$$A_1 = xy = x^1 x^2$$

$$A_2 = 2y - z^2 = 2x^2 - \left(x^3\right)^2$$

$$A_3 = xz = x^1 x^3$$

where care must be taken to distinguish between superscripts and exponents.

Let $\overline{A}_k$ denote the covariant components in spherical coordinates

$$\overline{x}^1 = r, \overline{x}^2 = \theta, \overline{x}^3 = \phi$$

Then

$$\overline{A}_k = \frac{\partial x^j}{\partial \overline{x}^k} A_j \cdots\cdots\cdots (1)$$

The transformation equations between coordinate systems are

$$x^1 = \overline{x}^1 \sin \overline{x}^2 \cos \overline{x}^3$$

$$x^2 = \overline{x}^1 \sin \overline{x}^2 \cos \overline{x}^3$$

$$x^3 = \overline{x}^1 \cos \overline{x}^2$$

Then equation (1) yield the required covariant components

$$\overline{A}_1 = \frac{\partial x^1}{\partial \overline{x}^1} A_1 + \frac{\partial x^2}{\partial \overline{x}^1} A_2 + \frac{\partial x^3}{\partial \overline{x}^1} A_3$$

$$= \left(\sin \overline{x}^{-2} \cos \overline{x}^3\right)\left(x^1 x^3\right) + \left(\sin \overline{x}^2 \sin \overline{x}^3\right)\left(2x^2 - x^3\right)^2$$

$$+ \left(\cos \overline{x}^2\right)\left(x^1 x^3\right)$$

$$= (\sin\theta \cos\phi)(xy) + (\sin\theta\cos\phi)\left(2y - z^2\right)$$

$$+ \cos\theta\,(xy)$$

$$= (\sin\theta\cos\phi)\left(r^2 \sin^2\theta \sin\phi\cos\phi\right)$$

$$+ (\sin\theta\sin\phi)\left(2r\sin\theta\sin\phi - r^2\cos^2\theta\right)$$

$$+ (\cos\theta)\left(r^2 \sin\theta\cos\theta\cos\phi\right)$$

$$\overline{A}_2 = \frac{\partial x^1}{\partial \overline{x}^2} A_1 + \frac{\partial x^2}{\partial \overline{x}^2} A_2 + \frac{\partial x^3}{\partial \overline{x}^2} A_3$$

$$= (r\cos\theta\cos\phi)\left(r^2\sin^2\theta\sin\phi\cos\phi\right)$$

$$+ (r\cos\theta\sin\phi)\left(2r\sin\theta\sin\phi - r^2\cos^2\theta\right)$$

$$+ (-r\sin\theta)\left(r^2\sin\theta\cos\theta\cos\phi\right)$$

$$\overline{A}_3 = \frac{\partial x^1}{\partial \overline{x}^3} A_1 + \frac{\partial x^2}{\partial \overline{x}^3} A_2 + \frac{\partial x^3}{\partial \overline{x}^3} A_3$$

$$= (-r\sin\theta\sin\phi)\left(r^2\sin^2\theta\sin\phi\cos\phi\right)$$

$$+ (r\sin\theta\cos\phi)\left(2r\sin\theta\sin\phi - r^2\cos^2\theta\right)$$

$$+ (0)\left(r^2\sin\theta\cos\theta\cos\phi\right)$$

Example (3):

Show that

i. $\dfrac{\partial x^p}{\partial x^q} = \delta_p^q,$

ii. $\dfrac{\partial x^p}{\partial \overline{x}^q}\dfrac{\partial \overline{x}^q}{\partial x^r} = \delta_r^p$

(i) if $p = q, \dfrac{\partial x^p}{\partial x^q} = 1$ since $x^p = x^q$

if $p \neq q, \dfrac{\partial x^p}{\partial x^q} = 0$

since x^p and x^q are independant

then $\dfrac{\partial x^p}{\partial x^q} = \delta_p^q$

(ii) Coordinates x^p are functions of coordinates $\overline{x}^q$ which are in turn functions of coordinates x^r. Then by the chain rule we get

$$\frac{\partial x^p}{\partial x^r} = \frac{\partial x^p}{\partial \overline{x}^q} \cdot \frac{\partial \overline{x}^q}{\partial x^r} = \delta_r^p$$

Example (4):

If $\overline{A}^p = \dfrac{\partial \overline{x}^p}{\partial x^q} A_q$ prove that $A^q = \dfrac{\partial x^q}{\partial \overline{x}^p} \overline{A}^p$.

Since

$$\overline{A}^p = \frac{\partial \overline{x}^p}{\partial x^q} \overline{A}^q \Rightarrow \overline{A}^p \frac{\partial \overline{x}^r}{\partial x^q} = \frac{\partial x^r}{\partial \overline{x}^p} \cdot \frac{\partial \overline{x}^p}{\partial x^q} A^q = \delta_q^r A^q = A$$

Replacing r & q we get

$$A^q = \frac{\partial x^q}{\partial \overline{x}^p} \overline{A}^p$$

Example (5):

Prove that $\dfrac{\partial A_p}{\partial x^q}$ is not a tensor even though A_p is a covariant tensor of rank one.

Since A_p is a tensor then

$$\overline{A}_j = \frac{\partial x^p}{\partial \overline{x}^j} A_p$$

Differentiating with respect to $\overline{x}^k$.

$$\frac{\partial \overline{A}_j}{\partial \overline{x}^k} = \frac{\partial x^p}{\partial \overline{x}^j} \frac{\partial A_p}{\partial \overline{x}^k} + \frac{\partial^2 x^p}{\partial \overline{x}^k \partial \overline{x}^j} A_p$$

$$= \frac{\partial x^p}{\partial \overline{x}^j} \frac{\partial A_p}{\partial x^q} \frac{\partial x^q}{\partial \overline{x}^k} + \frac{\partial^2 x^p}{\partial \overline{x}^k \partial \overline{x}^j} A_p$$

$$= \frac{\partial x^p}{\partial \overline{x}^j} \frac{\partial x^q}{\partial \overline{x}^k} \frac{\partial A_p}{\partial x^q} + \frac{\partial^2 x^p}{\partial \overline{x}^k \partial \overline{x}^j} A_p$$

Since the second term on the right is present,

$\frac{\partial A_p}{\partial x^q}$ does not transform as a tensor should.

Later we shall show how the addition of suitable quantity to $\frac{\partial A_p}{\partial x^q}$ occurs the result to be a tensor.

Example (6):

Evaluate (a) $\delta_p^q A_s^{qr}$, (b) $\delta_q^p \delta_r^q$.

Since $\delta_q^p = 1$ if $p = q$ and 0 if $p \neq q$,

We have (a) $\delta_p^q \delta_s^{qr} = A_s^{pr}$

(b) $\delta_q^p \delta_r^q = \delta_r^p$

Example (7):

Prove that δ_q^p is mixed tensor of the second rank.

Since

$$\delta_q^p = \frac{\partial x^p}{\partial x^q} = \begin{cases} 1 & p = q \\ 0 & p \neq q \end{cases}$$

And

$$\bar{\delta}_k^j = \frac{\partial \bar{x}^j}{\partial \bar{x}^k} = \begin{cases} 1 & j = k \\ 0 & j \neq k \end{cases}$$

Hence

$$\bar{\delta}_k^j = \frac{\partial \bar{x}^j}{\partial \bar{x}^k} = \frac{\partial \bar{x}^j}{\partial x^p} \cdot \frac{\partial x^p}{\partial \bar{x}^k} = \frac{\partial x^j}{\partial x^p} \cdot \frac{\partial x^p}{\partial x^q} \cdot \frac{\partial x^q}{\partial \bar{x}^k}$$

$$\bar{\delta}_k^j = \frac{\partial \bar{x}^j}{\partial x^p} \cdot \frac{\partial x^q}{\partial \bar{x}^k} \cdot \frac{\partial x^p}{\partial x^q} = \frac{\partial \bar{x}^j}{\partial x^p} \cdot \frac{\partial x^q}{\partial \bar{x}^k} \delta_q^p$$

it follows that δ_q^p is a mixed tensor of rank two.

Example (8):

Show that every tensor can be expressed as the sum of two tensors. One of which is symmetric and the other skew-symmetric.

Let B^{pq} is a tensor, then

$$\begin{aligned} B^{pq} &= \frac{1}{2}\left(B^{pq} + B^{qp}\right) + \frac{1}{2}\left(B^{pq} - B^{qp}\right) \\ &= R^{pq} + S^{pq} \end{aligned}$$

where

$$R^{pq} = \frac{1}{2}\left(B^{pq} - B^{qp}\right) = -R^{qp}$$

$$S^{pq} = \frac{1}{2}\left(B^{pq} + B^{qp}\right) = S^{pq}$$

i.e. R^{pq} is skew-symmetric, and S^{pq} is symmetric tensor.

Example (9):

Express in matrix notation the transformation equation for a covariant vector and contra-variant tensor of rank two assuming $N = 3$.

The transformation equations

$\overline{A}_p = \dfrac{\partial \overline{x}^q}{\partial \overline{x}^p} A_q$ can be written as

$$\begin{bmatrix} \overline{A}_1 \\ \overline{A}_2 \\ \overline{A}_3 \end{bmatrix} = \begin{bmatrix} \dfrac{\partial x^1}{\partial \overline{x}^1} & \dfrac{\partial x^2}{\partial \overline{x}^1} & \dfrac{\partial x^3}{\partial \overline{x}^1} \\ \dfrac{\partial x^1}{\partial \overline{x}^2} & \dfrac{\partial x^2}{\partial \overline{x}^2} & \dfrac{\partial x^3}{\partial \overline{x}^2} \\ \dfrac{\partial x^1}{\partial \overline{x}^3} & \dfrac{\partial x^2}{\partial \overline{x}^3} & \dfrac{\partial x^3}{\partial \overline{x}^3} \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix}$$

In terms of column vectors, or equivalently in terms of row vectors.

$$\begin{pmatrix} \overline{A}_1 & \overline{A}_2 & \overline{A}_3 \end{pmatrix} = \begin{pmatrix} A_1 & A_2 & A_3 \end{pmatrix} \begin{bmatrix} \dfrac{\partial x^1}{\partial \overline{x}^1} & \dfrac{\partial x^1}{\partial \overline{x}^2} & \dfrac{\partial x^1}{\partial \overline{x}^3} \\ \dfrac{\partial x^2}{\partial \overline{x}^1} & \dfrac{\partial x^2}{\partial \overline{x}^2} & \dfrac{\partial x^2}{\partial \overline{x}^3} \\ \dfrac{\partial x^3}{\partial \overline{x}^1} & \dfrac{\partial x^3}{\partial \overline{x}^2} & \dfrac{\partial x^3}{\partial \overline{x}^3} \end{bmatrix}$$

the transformation equation for contra-variant tensor is

$$\overline{A}^{pq} = \frac{\partial \overline{x}^p}{\partial x^r} \frac{\partial \overline{x}^q}{\partial x^s} A^{rs}$$

can be written

$$\begin{bmatrix} \overline{A}^{11} & \overline{A}^{12} & \overline{A}^{13} \\ \overline{A}^{21} & \overline{A}^{22} & \overline{A}^{23} \\ \overline{A}^{31} & \overline{A}^{32} & \overline{A}^{33} \end{bmatrix} = \begin{bmatrix} \frac{\partial x^1}{\partial \overline{x}^1} & \frac{\partial x^1}{\partial \overline{x}^2} & \frac{\partial x^1}{\partial \overline{x}^3} \\ \frac{\partial x^2}{\partial \overline{x}^1} & \frac{\partial x^2}{\partial \overline{x}^2} & \frac{\partial x^2}{\partial \overline{x}^3} \\ \frac{\partial x^3}{\partial \overline{x}^1} & \frac{\partial x^3}{\partial \overline{x}^2} & \frac{\partial x^3}{\partial \overline{x}^3} \end{bmatrix} \begin{bmatrix} A^{11} & A^{12} & A^{13} \\ A^{21} & A^{22} & A^{23} \\ A^{31} & A^{32} & A^{33} \end{bmatrix} \cdot \begin{bmatrix} \frac{\partial x^1}{\partial \overline{x}^1} & \frac{\partial x^2}{\partial \overline{x}^1} & \frac{\partial x^3}{\partial \overline{x}^1} \\ \frac{\partial x^1}{\partial \overline{x}^2} & \frac{\partial x^2}{\partial \overline{x}^2} & \frac{\partial x^3}{\partial \overline{x}^2} \\ \frac{\partial x^1}{\partial \overline{x}^3} & \frac{\partial x^2}{\partial \overline{x}^3} & \frac{\partial x^3}{\partial \overline{x}^3} \end{bmatrix}$$

Extensions of these results can be made for $N > 3$. For higher rank tensors, however, the matrix notation fails.

Fundamental Operations with Tensors:

1. Addition:

The sum of two or more tensors of the same rank and type (i.e. same number of contra-variant indices and same number of covariant indices) is also a tensor of the same rank and type. Thus if A_q^{mp} and B_q^{mp} are tensors, then

$$C_q^{m\,p} = A_q^{mp} + B_q^{mp}$$

is also a tensor.

Addition of tensors is cumulative and

associative.

$$A_q^{mp} + B_q^{mp} = B_q^{mp} + A_q^{mp}$$

$$A_q^{mp} + \left(3_q^{mp} + C_q^{mp}\right) = \left(A_q^{mp} + B_q^{mp}\right) + C_q^{mp}$$

2. Subtraction:

The difference of two tensors of the same rank and type is also a tensor of the same rank and type. Thus if A_q^{mp} and B_q^{mp} are tensors, then

$$D_q^{m\ p} = A_q^{mp} - B_q^{mp}$$

is also a tensor.

3. Outer Multiplication:

The product of two tensors is a tensor whose rank is the sum of the ranks of the given tensors. This product which involves ordinary multiplication of the components of tensor is called the outer product. For example,

$$A_q^{pr} B_s^m = C_{qs}^{prm}$$

is the outer product of A_q^{pr} and B_s^m. However, note that not every tensor can be written as product of two tensors of lower rank.
For this reason division of tensors is not always possible.

4. Contraction:

Let us start with any mixed tensor, say $A_{l\,mn}^{ij}$,

and form the sum A^{ij}_{lmj}. In this case one contra-variant and one covariant index of the considered tensor are set equal $(j = n)$. We then have such a transformation relations.

$$\overline{A}^{st}_{pqr} = \frac{\partial \overline{x}^s}{\partial x^i}\frac{\partial \overline{x}^t}{\partial x^j}\frac{\partial x}{\partial \overline{x}^p}\frac{\partial x^m}{\partial \overline{x}^q}\frac{\partial x^n}{\partial \overline{x}^r}A^{ij}_{lmn},$$

$$\overline{A}^{sr}_{pqr} = \frac{\partial \overline{x}^s}{\partial x^i}\frac{\partial \overline{x}^r}{\partial x^j}\frac{\partial x}{\partial \overline{x}^p}\frac{\partial x^m}{\partial \overline{x}^q}\frac{\partial x^n}{\partial \overline{x}^r}A^{ij}_{lmn}$$

$$= \frac{\partial \overline{x}^s}{\partial x^i}\frac{\partial x}{\partial \overline{x}^p}\frac{\partial x^m}{\partial \overline{x}^q}\delta^n_j A^{ij}_{lmn}$$

$$= \frac{\partial \overline{x}^s}{\partial x^i}\frac{\partial x^l}{\partial \overline{x}^p}\frac{\partial x^m}{\partial \overline{x}^q}\delta^n_j A^{in}_{lmn}$$

Thus we observe that A^{ij}_{lmn} is a mixed tensor, contra-variant of first order and covariant of the second order. This process, which is called counteraction, enables us to obtain a tensor of order 2 from a mixed tensor of order r. In the above example we could contract a stage further and arrive at the covariant vector A^{ij}_{lmn}. When contracting, any superscript may be used to sum with any subscript. Therefore we can form the following different tensors by contraction:

$$A^{ij}_{lmn}, A^{ij}_{ljn}, A^{ij}_{jmn}, A^{ij}_{lmi}, A^{ij}_{lin}, A^{ij}_{imn},$$

$$A^{ij}_{lij}, A^{ij}_{lji}, A^{ij}_{imi}, A^{ij}_{ijn}, A^{ij}_{jin}.$$

If the tensor A^{ij}_{lmn} possesses any symmetric properties, there will be fewer tensors formed from it by contraction.

As another example, the invariant A^i_i is formed by counteraction from the mixed tensor A^i_j. This justifies us in calling an invariant a tensor of zero order.

5. Inner Multiplication:

By the process of outer multiplication of two tensors followed by counteraction, we obtain a new tensor called an inner product of the given tensors. This process is called inner multiplication. For example, given tensor A^{mp}_q and B^r_{st}, the outer product is $A^{mp}_q B^r_{st}$. Letting $q = r$, we obtain the inner product $A^{mp}_q B^r_{st}$. Letting $q = r$ and $p = s$, another inner product $A^{mp}_r B^r_{pt}$ is obtained. Inner and outer multiplication of tensors is cumulative and associative.

6. Quotient Law:

Suppose it is not known whether a quantity X is a tensor or not. If an inner product of X with an arbitrary tensor is itself a tensor, then X is also a tensor. This is called the quotient law.

It is sufficient to consider the proof for the following particular case. The set of N^3 functions A^{ijk} form the components of tensor of the type indicated by its indices if a relation such as:

$$A^{ijk} A^p_{ij} = C^{pk}, \text{ holds.}$$

Provided that A^p_{ij} is arbitrary tensor and C^{pk} a tensor. The transformed quantities, referred to system of coordinate $\overline{x}^i$ satisfy the equations

$$\overline{A}^{ijk} \overline{B}^p_{ij} = \overline{C}^{pk}$$

which can be put in the form

$$\overline{A}^{ijk} \frac{\partial \overline{x}^p}{\partial x^l} \frac{\partial x^m}{\partial \overline{x}^i} \frac{\partial x^n}{\partial \overline{x}^j} B^l_{mn} = \frac{\partial \overline{x}^p}{\partial x^q} \frac{\partial \overline{x}^k}{\partial x^r} C^{qr}$$

$$= \frac{\partial \overline{x}^p}{\partial x^q} \frac{\partial \overline{x}^k}{\partial x^r} A^{ijr} B^q_{ij}$$

With a change of dummy indices, we get

$$\frac{\partial \overline{x}^p}{\partial x^l} \left\{ \overline{A}^{ijk} \frac{\partial x^m}{\partial \overline{x}^i} \frac{\partial x^n}{\partial \overline{x}^j} - A^{mnr} \frac{\partial \overline{x}^k}{\partial x^r} \right\} B^l_{nn} = 0$$

On inner multiplication by $\dfrac{\partial x^s}{\partial \overline{x}^p}$, we obtain.

$$\left\{ \overline{A}^{ijk} \frac{\partial x^m}{\partial \overline{x}^i} \frac{\partial x^n}{\partial \overline{x}^j} - A^{mnr} \frac{\partial \overline{x}^k}{\partial x^r} \right\} B^s_{mn} = 0$$

Since B^s_{mn} is an arbitrary tensor, we can

arrange that only one of its components differs from zero. Now each component of B^{s}_{mn} may be selected in turn as that one which does not vanish. This shows that the expression in brackets is identically zero. That is:

$$\overline{A}^{ijk}\frac{\partial x^m}{\partial \overline{x}^i}\frac{\partial x^n}{\partial \overline{x}^j} = A^{mnr}\frac{\partial \overline{x}^k}{\partial x^r}$$

Inner multiplication of this equation by $\frac{\partial \overline{x}^s}{\partial x^m}\frac{\partial \overline{x}^t}{\partial x^n}$ gives the result.

$$\overline{A}^{stk} = \frac{\partial \overline{x}^s}{\partial x^m}\frac{\partial \overline{x}^t}{\partial x^n}\frac{\partial \overline{x}^k}{\partial x^r}A^{mnr}$$

Thus A^{mnr} is a tensor of the third order and contra-variant in all its indices. This proves the quotiont law.

7. Derivatives:

We have considered $\phi = \phi\left(x^k\right)$ which is a scaler function of position. The coordinate derivatives of ϕ, i.e. $\frac{\partial \phi}{\partial x^k}$ form a covariant vector since.

$$\frac{\partial \phi}{\partial \overline{x}^k} = \frac{\partial \phi}{\partial x^i}\frac{\partial x^i}{\partial \overline{x}^k} = \frac{\partial x^i}{\partial \overline{x}^k}\frac{\partial \phi}{\partial x^i}$$

This can easily be seen. Differentiating the transformation law.

$$A^k = \overline{A}^i \frac{\partial x^k}{\partial \overline{x}^i}$$

(w.r.t) x^j and obtain

$$\frac{\partial A^k}{\partial x^j} = \frac{\partial \overline{A}^i}{\partial \overline{x}^n} \frac{\partial \overline{x}^n}{\partial x^j} \frac{\partial x^k}{\partial \overline{x}^i} + \overline{A}^i \frac{\partial^2 x^k}{\partial \overline{x}^i \partial \overline{x}^n} \frac{\partial \overline{x}^n}{\partial x^j}$$

The presence of the last term on the (r.h.s) of this equation shows that the partial derivatives $\frac{\partial A^k}{\partial x^j}$ do not form a tensor.

This type of differentiation is not accepted in this calculus since it is not an invariant operation, we shall define a new type of differentiation which is invariant under the transformation.

Riemann Space – The Line Element and Metric Tensor

The interval between any two neighboring points (x, y, z) and $(x + dx, y + dy, z + dz)$ in any Euclidian 3-dimentional space is given as $dS^2 = dx^2 + dy^2 + dz^2$ in rectangular coordinates.

$dS^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2$ in spherical coordinates.

We notice that the characteristics of

Euclidian geometry are:

(1) Each point in the space is defined by 3-coordinates.

(2) The interval between any two neighboring points is given by a certain quadratic expression in the increment of coordinates.

Riemann in defining his geometry has generalized these two characteristics in the following.

(1) Each point in the space is defined by N-coordinate $\left(x^1, x^2, \cdots, x^N\right)$.

(2) The interval between any two neighboring points or the line element dS in this space is given by the quadratic form, called the metric form or metric,

$$dS^2 = \sum_{p=1}^{N}\sum_{q=1}^{N} g_{pq} dx^p\, dx^q \quad , p, q = 1, 2, \cdots, N$$

Or using the summation convention

$$dS^2 = g_{pq} dx^p\, dx^q$$

The quantities g_{pq} are the components of a covariant tensor or rank two called the metric

tensor of fundamental tensor. We can always will choose this tensor to be symmetric.

Example [1]

If $\phi = a_{jk}A^j A^k$ show that we can always write

$\phi = b_{jk}A^j A^k$ where b_{jk} is symmetric.

$$\phi = b_{jk}A^j A^k = a_{jk}A^k A^j = a_{kj}A^j A^k$$

Then

$$2\phi = a_{jk}A^j A^k + a_{kj}A^j A^k = \left(a_{jk} + a_{kj}\right)A^j A^k$$

and

$$\phi = \frac{1}{2}\left(a_{jk} + a_{kj}\right)A^j A^k = b_{jk}A^j A^k$$

where

$$b_{jk} = \frac{1}{2}\left(a_{jk} + a_{kj}\right) = b_{kj} \text{ is symmetric.}$$

Example [2]

If $ds^2 = g_{jk}dx^j dx^k$ is an invariant, show that g_{jk} is symmetric covariant tensor of rank two.

By ex.1 $\phi = ds^2, A^j = dx^j$ and $A^k = dx^k$; it follows that g_{jk} can be chosen symmetric.

Also since ds^2 is an invariant,

$$\overline{g}_{pq} d\overline{x}^p d\overline{x}^q = q_{jk} dx^j dx^k = g_{jk} \frac{\partial x^j}{\partial \overline{x}^p} d\overline{x}^p \frac{\partial x^k}{\partial \overline{x}^q} d\overline{x}^q$$

$$= g_{jk} \frac{\partial x^j}{\partial \overline{x}^p} \frac{\partial x^k}{\partial \overline{x}^q} d\overline{x}^p d\overline{x}^q$$

then $\overline{g}_{pq} = g_{jk} \dfrac{\partial x^j}{\partial \overline{x}^p} \dfrac{\partial x^k}{\partial \overline{x}^q}$ and g_{jk} is symmetric covariant tensor of rank two, called the metric tensor.

Example [3]

(a) Express the determinant

$$g = \begin{vmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{vmatrix}$$ in terms of the

elements in the second row and their corresponding cofactors.

(b) Show that $g_{jk} G(j,k) = g$ where $G(j,k)$ is the cofactor of g_{jk} in g and where summation is over k only.

(a) The cofactor of g_{jk} is the determinant obtained from g by:

(i) deleting the raw and column in which g_{jk} appears and,

(ii) associating the sign $(-1)^{j+k}$ to this determinant.

Thus cofactor of

$$g_{21} = (-1)^{2+1}\begin{vmatrix} g_{12} & g_{13} \\ g_{32} & g_{33} \end{vmatrix}$$

cofactor of

$$g_{22} = (-1)^{2+2}\begin{vmatrix} g_{11} & g_{13} \\ g_{31} & g_{33} \end{vmatrix}$$

cofactor of

$$g_{23} = (-1)^{2+3}\begin{vmatrix} g_{11} & g_{12} \\ g_{31} & g_{32} \end{vmatrix}$$

Denoting these cofactors by $G(2,1), G(2,2)$ and $G(2,3)$ respectively. Then by an elementary principle of determinants.

(b) By appling the result of (a) to any raw or column, we have $g_{jk}G(j,k) = g$ where the summation is over k only. These results hold where $g = |g_{jk}|$ is an N^{th} order determinant.

Example [4]

(a) Prove that

$g_{21}G(3,1) + g_{22}G(3,2) + g_{23}G(3,3) = 0$.

(b) Prove that $g_{jk}G(p,k)=0$ if $j\neq p$.

(a) Consider the determinant $\begin{vmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{21} & g_{22} & g_{23} \end{vmatrix}$

which is zero, its last two rows are identical. Expanding according to elements of last row we have,

$$g_{21}G(3,1)+g_{22}G(3,2)+g_{23}G(3,3)=0.$$

(b) By setting the corresponding elements of any two rows (or columns) equal, we can show, as in part (a), that $g_{jk}G(p,k)=0$ if $j\neq p$. This result holds for N^{th} order determinants as well.

Conjugate or Reciprocal Tensors:

Let $g=\left|g_{pq}\right|$ denotes the determinant with elements g_{pq} and suppose $g\neq 0$. Define g^{pq} by:

$$g^{pq}=\frac{\text{cofactor of } g_{pq}}{g}$$

Then g^{pq} is a symmetric contra-variant tensor

of rank two called the conjugate or reciprocal tensor of g_{pq}. It can be shown that:

$$g^{pq}g_{rq} = \delta_r^p$$

Example [5]

Define $g^{jk} = \dfrac{G(j,k)}{g}$ where $G(j,k)$ is the cofactor of g_{jk} in the determinant $g = |g_{jk}| \neq 0$.

Prove that $g_{jk}g^{pk} = \delta_j^p$.

By ex.3 we have

$$g_{jk}G(j,k) = g$$

$$or \ g_{jk}\frac{G(j,k)}{g} = 1$$

$$or \ g_{jk}g^{jk} = 1$$

where the summation is over k only.

By ex.4 we have

$$g_{jk}G(p,k) = 0$$

$$or \ g_{jk}\frac{G(p,k)}{g} = 0$$

$$or \ g_{jk}g^{pk} = 0 \quad \text{if } p \neq j$$

Then

$$g_{jk}g^{pk}\left\{= 1 \text{ if } p = j, \text{ and } 0 \text{ if } p \neq j\right\} = \delta_j^p.$$

Example [6]

Prove that g^{jk} is a symmetric contra-variant tensor of rank two.

Since g_{jk} is symmetric, $G(j,k)$ is symmetric and so $g^{ik} = \dfrac{G(j,k)}{g}$ is symmetric.

If B^p is an arbitrary contra-variant vector, $B_q = g_{pq}B^p$ is an arbitrary covariant. Multiplying by g^{jq},

$$g^{jq}B_q = g^{jq}g_{pq}B^p = \delta_p^j B^p = B^j$$

or

$$g^{jp}B_q = B^j$$

Since B_q is an arbitrary vector, g^{jq} is then a contra-variant tensor of rank two, by application of the quotient variant law. The tensor g^{jk} is called the conjugate metric tensor.

ASSOCIATED TENSORS

Given a tensor, we can derive other tensors by raising or lowering indices. For example, given the tensor A_{pq} we obtain by raising the index p.The tensor $A^p_{\bullet q}$, the dot indicating the original position of the moved index. By raising the

index q also we obtain $A_{\bullet\bullet}^{pq}$. Where no confusion can arise we shall often omit the dots; thus $A_{\bullet\bullet}^{pq}$ can be written $A^{pq}{}_{\bullet}$. These derived tensors can be obtained by forming inner products of the given tensor with the metric tensor g_{pq} or its conjugate g^{pq}. Thus for example,

$$A_{\bullet q}^{p} = g^{rp} A_{qq}$$

$$A^{pq} = g^{rp} g^{sp} A_{rs},$$

$$A_{\bullet rs}^{p} = g_{rq} A_{\bullet\bullet s}^{pq},$$

$$A_{\bullet\bullet n}^{qm\bullet tk} = g^{pk} g_{sn} g^{rm} A_{\bullet r\bullet\bullet p}^{q\bullet st}$$

These become clean if we interpret multiplication by g^{rp} as meaning let $r = p$ (or $p = r$) in whatever follows and raise index. Similarly we interpret multiplication by g_{rq} as meaning; let $r = q$ (or $q = r$) in whatever follows and lower this index.

All tensors obtained from a given tensor by forming inner products with the metric tensor and its conjugate are celled associated tensors of the given tensor. For example A^{m} anu

are associated tensors, the first are contra-

variant and the second covariant components. The relation between them is given by.

$$A_p = g_{pq} A^q \quad or \quad A^p = g^{pq} A_q$$

Geodesics

The distance S between two points t_1 and t_2 on a curve $x^r = x^r(t)$ in a Riemannian space is given by:

$$S = \int_{t_1}^{t_2} \sqrt{g_{pq} \frac{dx^p}{dt} \frac{dx^q}{dt}} \cdot dt$$

That curve in space which makes the distance a minimum is called a geodesic of the space. In what follows we shall make use of the calculus of variations ((ex7) and (ex8)) to show that the geodesics are found from the differential equation:

$$\frac{d^2 x^r}{ds^2} + \begin{Bmatrix} r \\ p \quad q \end{Bmatrix} \frac{dx^p}{ds} \frac{dx^q}{ds} = 0$$

where s is the arc length parameter.
Here

$$\begin{Bmatrix} r \\ p \quad q \end{Bmatrix} = g^{rk} \left[pq, k \right],$$

$$\left[pq, k \right] = \frac{1}{2} \left(\frac{\partial g_{pk}}{\partial x^q} + \frac{\partial g_{qk}}{\partial x^p} - \frac{\partial g_{pq}}{\partial x^k} \right)$$

and the two symbols $[pq,k]$ and $\left\{\begin{matrix} r \\ p\ \ q \end{matrix}\right\}$ are called Christoffel's symbols of the first and of the second kind respectively.

Example [7]

Prove that a necessary condition that

$$I = \int_{t_1}^{t_2} F(t,x,\dot{x})\,dt$$

be an extremum (maximum or minimum) is that

$$\frac{\partial F}{\partial x} - \frac{d}{dt}\left(\frac{\partial F}{\partial x}\right) = 0$$

Let the curve which makes I an extremum be

$$x = X(t) \qquad , \qquad t_1 < t < t_2 \ .$$

Then

$$x = X(t) + \varepsilon\eta(t),$$

where ε is independent of t is a neighboring curve through t_1 and t_2 so that

$\eta(t_1) = \eta(t_2) = 0$

the value of I for the neighboring curve is

$I(\varepsilon) = \int_{t_1}^{t_2} F(t, X + \varepsilon\eta, X + \varepsilon\eta)\,dt$

This is an extremum for $\varepsilon = 0$. A necessary condition that this be so is that $\left.\frac{dI}{d\varepsilon}\right|_{\varepsilon=0} = 0$. But

by differentiation under the integral sign, assuming this valid,

$$\left.\frac{dI}{d\varepsilon}\right|_{\varepsilon=0} = \int_{t_1}^{t_2}\left(\frac{\partial F}{\partial x}\cdot\frac{\partial x}{\partial \varepsilon} + \frac{\partial F}{\partial x}\cdot\frac{\partial \dot{x}}{\partial \varepsilon}\right)dt$$

$$= \int_{t_1}^{t_2}\left(\frac{\partial F}{\partial x}\eta + \frac{\partial F}{\partial \dot{x}}\eta\right)dt = 0$$

which can be written as

$$\int_{t_1}^{t_2}\frac{\partial F}{\partial x}\eta\,dt + \left.\frac{\partial F}{\partial \dot{x}}\eta\right|_{t_1}^{t_2} - \int_{t_1}^{t_2}\eta\frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}}\right)dt$$

$$= \int_{t_1}^{t_2}\eta\left\{\frac{\partial F}{\partial x} - \frac{d}{dt}\left(\frac{\partial F}{\partial x}\right)\right\}dt = 0$$

since η is arbitrary, the integrand

$$\frac{\partial F}{\partial x} - \frac{d}{dt}\left(\frac{\partial F}{\partial x}\right) = 0.$$

The result is easily extended to the integral

$$\int_{t_1}^{t_2} F\left(t, x^1, x^1, x^2, x^2, \cdots, x^N, x^N\right)dt$$

and gives

$$\frac{\partial F}{\partial x^k} - \frac{d}{dt}\left(\frac{\partial F}{\partial x^k}\right) = 0,$$

called Euler's or Lagrange,s equations.

Example [8]

Let us show that the geodesics in Riemannian space are given by:

$$\frac{d^2x^r}{ds^2} + \begin{Bmatrix} r \\ p \quad q \end{Bmatrix} \frac{dx^p}{ds}\frac{dx^q}{ds} = 0$$

we must determine the extremum of

$$\int_{t_2}^{t_1} \sqrt{g_{pq} x^{\bullet p} x^{\bullet q}} \cdot dt$$

using Euler's equations (ex7) with

$$F = \sqrt{g_{pq} x^{\bullet p} x^{\bullet q}}.$$

We have

$$\frac{\partial F}{\partial x^k} = \frac{1}{2} \cdot \frac{1}{\sqrt{g_{pq} x^{\bullet p} x^{\bullet q}}} \cdot \frac{\partial g_{pq}}{\partial x^k} x^{\bullet p} x^{\bullet q}$$

$$= \frac{1}{2s^\bullet} \frac{\partial g_{pq}}{\partial x^k} x^{\bullet p} x^{\bullet q}$$

$$\frac{\partial F}{\partial x^{\bullet k}} = \frac{1}{2\sqrt{g_{pq} x^{\bullet p} x^{\bullet q}}} \cdot \frac{\partial}{\partial x^{\bullet k}} \left\{ g_{pq} x^{\bullet p} x^{\bullet q} \right\}$$

$$= \frac{1}{2s^\bullet} g_{pq} \left\{ x^{\bullet p} \frac{\partial}{\partial x^{\bullet k}} x^{\bullet q} + x^{\bullet q} \frac{\partial}{\partial x^{\bullet k}} x^{\bullet p} \right\}$$

$$= \frac{1}{2s^\bullet} g_{pq} \left\{ x^{\bullet p} \delta_{qk} + x^{\bullet q} \delta_{pk} \right\}$$

$$\frac{\partial F}{\partial x^{\bullet k}} = \frac{1}{2s^{\bullet}}\left\{g_{pk}x^{\bullet p} + g_{kq}x^{\bullet q}\right\}$$
$$= \frac{1}{s^{\bullet}}g_{pk}x^{\bullet p}$$

Now applying condition

$$\frac{\partial F}{\partial x^{k}} - \frac{d}{dt}\left(\frac{\partial F}{\partial x^{\bullet k}}\right) = 0$$

we shall have:

$$\frac{1}{2s^{\bullet}}\frac{\partial g_{pq}}{\partial x^{k}}x^{\bullet p}x^{\bullet q} = \frac{s^{\bullet}\left\{g_{pk}x^{\bullet\bullet p} - \frac{\partial g_{pk}}{\partial x^{q}}x^{\bullet p}x^{\bullet q}\right\} - g_{pk}x^{\bullet p}s^{\bullet\bullet}}{s^{\bullet 2}}$$

Or

$$\frac{1}{2}\frac{\partial g_{pq}}{\partial x^{k}}x^{\bullet p}x^{\bullet q} - g_{pk}x^{\bullet\bullet p} - \frac{\partial g_{pk}}{\partial x^{q}}x^{\bullet p}x^{\bullet q} = -\frac{1}{s^{\bullet}}g_{pk}x^{\bullet p}s^{\bullet\bullet}$$

Writing

$$\frac{\partial g_{pk}}{\partial x^{q}}x^{\bullet p}x^{\bullet q} = \frac{1}{2}\left\{\frac{\partial g_{pk}}{\partial x^{q}} + \frac{\partial g_{qk}}{\partial x^{p}}\right\}x^{\bullet p}x^{\bullet q},$$

Then equation (1) becomes.

$$-\frac{1}{2}\frac{\partial g_{pq}}{\partial x^{k}}x^{\bullet p}x^{\bullet q} + g_{pk}x^{\bullet\bullet p} + \frac{1}{2}\left\{\frac{\partial g_{pk}}{\partial x^{q}} + \frac{\partial g_{qk}}{\partial x^{p}}\right\}x^{\bullet p}x^{\bullet q}$$
$$= \frac{g_{pk}}{s^{\bullet}}x^{\bullet p}s^{\bullet\bullet}$$

or

$$g_{pk}x^{\bullet\bullet p} + \frac{1}{2}\left\{\frac{\partial g_{pk}}{\partial x^{q}} + \frac{\partial g_{qk}}{\partial x^{p}} - \frac{\partial g_{pq}}{\partial x^{k}}\right\}x^{\bullet p}x^{\bullet q} = \frac{g_{pk}x^{\bullet p}s^{\bullet\bullet}}{s^{\bullet}}$$

denoting

$$\frac{1}{2}\left\{\frac{\partial g_{pk}}{\partial x^q}+\frac{\partial g_{qk}}{\partial x^p}-\frac{\partial g_{pq}}{\partial x^k}\right\}=[pq,k],$$

and using arc length as parameter,
$s^{\bullet}=1, s^{\bullet\bullet}=0$ and eq.(2) becomes:

$$g_{pk}\frac{d^2x^p}{ds^2}+[pq,k]\frac{dx^p}{ds}\frac{dx^q}{ds}=0\ldots\ldots\ldots\ldots\ldots\ldots[3]$$

Multiplying eq.(3) by g^{rk} we obtain

$$\frac{d^2x^r}{ds^2}+\left\{\begin{matrix} r \\ p\ \ q\end{matrix}\right\}\frac{dx^p}{ds}\frac{dx^q}{ds}=0$$

Example [9]

Prove that the Euclidean 3-dimensional geometry, a geodesic given a straight line.

The equation of a geodesic curve is given by

$$\frac{d^2x^\sigma}{ds^2}+\left\{\begin{matrix} \sigma \\ \mu\ \ \nu\end{matrix}\right\}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}=0\ldots\ldots\ldots\ldots\ldots\ldots\ldots[1]$$

In 3- dimensional Euclidean we have:

$$ds^2=dx^2+dy^2+dz^2=dx_1{}^2+dx_2{}^2+dx_3{}^2\ldots[2]$$

Therefore

$g_{11}=g_{22}=g_{33}=1$ and $g_{\mu\nu}=0$ for $\mu\neq\nu$

hence $\left\{\begin{matrix} \sigma \\ \mu\ \ \nu\end{matrix}\right\}=0$

equation (1) becomes

$$\frac{d^2x^\sigma}{ds^2}=0 \qquad ,\sigma=1,2,3$$

$$\therefore \frac{d^2x^1}{ds^2}=0 \text{ gives } \frac{dx^1}{ds}=const.=l$$

Also

$$\frac{dx^2}{ds}=m \qquad ,\frac{dx^3}{ds}=n$$

By integration we get:

$$x^1=l\,s+a \qquad or\ x=l\,s+a$$

$$x^2=m\,s+b \qquad or\ y=m\,s+b$$

$$x^3=n\,s+c \qquad or\ z=n\,s+c$$

$$\therefore \frac{x-a}{l}=\frac{y-b}{m}=\frac{z-c}{n}=s$$

There represent equation of straight line.

We can prove that

$$l^2+m^2+n^2=1$$

as ,from (2) we have

$$1=\left(\frac{dx^1}{ds}\right)^2+\left(\frac{dx^2}{ds}\right)^2+\left(\frac{dx^3}{ds}\right)^2=l^2+m^2+n^2$$

Properties of Christoffl's Symbols

$$1-\left[p\,q,r\right]=\left[q\,p,r\right]$$

$$2-\begin{Bmatrix} s \\ p\ q \end{Bmatrix}=\begin{Bmatrix} s \\ q\ p \end{Bmatrix}$$

$$3-\left[p\,q,k\right]=g_{ks}\begin{Bmatrix} s \\ p\ q \end{Bmatrix}$$

$$4-\frac{\partial g_{pq}}{\partial x^{m}}=\left[p\,m,q\right]+\left[q\,m,p\right]$$

$$5-\frac{\partial g^{pq}}{\partial x^{m}}=-g^{pn}\begin{Bmatrix} q \\ m\ n \end{Bmatrix}-g^{qn}\begin{Bmatrix} p \\ m\ n \end{Bmatrix}$$

$$6-\begin{Bmatrix} p \\ p\ q \end{Bmatrix}=\frac{\partial}{\partial x^{q}}\ln\sqrt{g}$$

The proof:

$$1-\left[p\,q,r\right]=\frac{1}{2}\left(\frac{\partial g_{pr}}{\partial x^{q}}+\frac{\partial g_{qr}}{\partial x^{p}}-\frac{\partial g_{pq}}{\partial x^{r}}\right)$$

$$=\frac{1}{2}\left(\frac{\partial g_{qr}}{\partial x^{p}}+\frac{\partial g_{pr}}{\partial x^{q}}-\frac{\partial g_{qp}}{\partial x^{r}}\right)$$

$$=\left[q\,p,r\right]$$

$$2-\begin{Bmatrix} s \\ p\ q \end{Bmatrix}=g^{sr}\left[p\,q,r\right]=g^{sr}\left[q\,p,r\right]=\begin{Bmatrix} s \\ q\ p \end{Bmatrix}$$

$$3 - g_{ks}\begin{Bmatrix} s \\ p\ q \end{Bmatrix} = g_{ks}g^{sr}\left[p\,q,r\right] = \delta_k^r\left[p\,q,r\right] = \left[p\,q,k\right]$$

$$\text{or } \left[p\,q,k\right] = g_{ks}\begin{Bmatrix} s \\ p\ q \end{Bmatrix} \text{ i.e. } \left[p\,q,r\right] = g_{rs}\begin{Bmatrix} s \\ p\ q \end{Bmatrix}$$

Note that multiplying $\left[p\,q,r\right]$ by g^{sr} has the effect of replacing r by s, raising this index and replacing square brackets by braces to yield $\begin{Bmatrix} s \\ p\ q \end{Bmatrix}$. Similarly, multiplying $\begin{Bmatrix} s \\ p\ q \end{Bmatrix}$ by g_{sr} or g_{rs} has the effect of replacing s by r, lowering this index and replacing braces by square brackets to yield $\left[p\,q,r\right]$.

$$4 - \left[p\,m,q\right] + \left[q\,m,p\right] = \frac{1}{2}\left(\frac{\partial g_{pq}}{\partial x^m} + \frac{\partial g_{mq}}{\partial x^p} - \frac{\partial g_{pm}}{\partial x^q}\right)$$

$$+\frac{1}{2}\left(\frac{\partial g_{qp}}{\partial x^m} + \frac{\partial g_{mp}}{\partial x^q} - \frac{\partial g_{qm}}{\partial x^p}\right)$$

$$= \frac{\partial g_{pq}}{\partial x^m}$$

$$5 - \frac{\partial}{\partial x^m}\left(g^{jk}g_{ij}\right) = \frac{\partial}{\partial x^m}\left(\delta_i^k\right) = 0$$

Then

$$g^{ik}\frac{\partial g_{ij}}{\partial x^m}+\frac{\partial g^{jk}}{\partial x^m}g_{ij}=0$$

Or

$$g_{ij}\frac{\partial g^{jk}}{\partial x^m}=-g^{jk}\frac{\partial g_{ij}}{\partial x^m}$$

Multiplying by g^{ir} then,

$$g^{ir}g_{ij}\frac{\partial g^{jk}}{\partial x^m}=-g^{ir}g^{jk}\frac{\partial g_{ij}}{\partial x^m}$$

$$i.e.\,\delta_j^r\frac{\partial g^{jk}}{\partial x^m}=-g^{ir}g^{jk}\left(\left[im,j\right]+\left[jm,i\right]\right)$$

$$or\;\frac{\partial g^{rk}}{\partial x^m}=-g^{ir}\begin{Bmatrix}k\\ i\;m\end{Bmatrix}-g^{jk}\begin{Bmatrix}r\\ j\;m\end{Bmatrix}$$

and the result follows on replacing r,k,i,j by p,q,n,n respectively.

$6-g=g_{jk}G(j,k)$, (sum over k only).

Since $G(j,k)$ doesn't contain g_{jk} explicitly,

$\dfrac{\partial g}{\partial g_{jr}}=G(j,r)$. Then, summing over j

and r,

$$\frac{\partial g}{\partial x^m}=\frac{\partial g}{\partial g_{jr}}\frac{\partial g_{jr}}{\partial x^m}=G(j,r)\frac{\partial g_{jr}}{\partial x^m}$$

$$\frac{\partial g}{\partial x^m}=g\,g^{jr}\frac{\partial g_{jr}}{\partial x^m}=g\,g^{jr}\left(\left[j\,m,r\right]+\left[r\,m,j\right]\right)$$

$$\frac{\partial g}{\partial x^m}=g\left(\left\{\begin{matrix} j \\ j\ \ m\end{matrix}\right\}+\left\{\begin{matrix} r \\ r\ \ m\end{matrix}\right\}\right)=2g\left\{\begin{matrix} j \\ j\ \ m\end{matrix}\right\}$$

Then,

$$\frac{1}{2g}\frac{\partial g}{\partial x^m}=\left\{\begin{matrix} j \\ j\ \ m\end{matrix}\right\}\quad or\quad \left\{\begin{matrix} j \\ j\ \ m\end{matrix}\right\}=\frac{\partial}{\partial x^m}\ln\sqrt{g}$$

Example [1]

Drive transformation laws for Christoffel's symbols of (a) the first kind, (b) the second kind.

(a) Since $\overline{g}_{jk}=\dfrac{\partial x^p}{\partial \overline{x}^j}\dfrac{\partial x^q}{\partial \overline{x}^k}g_{pq}$

$$(1)\frac{\partial \overline{x}_{jk}}{\partial \overline{x}^m}=\frac{\partial x^p}{\partial \overline{x}^j}\frac{\partial x^q}{\partial \overline{x}^k}\frac{\partial x_{pq}}{\partial \overline{x}^r}\frac{\partial x^r}{\partial \overline{x}^m}+\frac{\partial x^p}{\partial \overline{x}^j}\frac{\partial^2}{\partial \overline{x}^m}\frac{x^q}{\partial \overline{x}^k}g_{pq}$$
$$+\frac{\partial^2}{\partial \overline{x}^m}\frac{x^p}{\partial \overline{x}^j}\frac{\partial x^q}{\partial \overline{x}^k}g_{pq}$$

By cyclic permutation or indices j,k,m and p,q,r.

$$(2)\frac{\partial \overline{x}_{km}}{\partial \overline{x}^j} = \frac{\partial x^q}{\partial \overline{x}^j}\frac{\partial x^r}{\partial \overline{x}^k}\frac{\partial x_{qr}}{\partial \overline{x}^r}\frac{\partial x^p}{\partial \overline{x}^m} + \frac{\partial x^q}{\partial \overline{x}^k}\frac{\partial^2}{\partial \overline{x}^j}\frac{x^r}{\partial \overline{x}^m}g_{qr} + \frac{\partial^2}{\partial \overline{x}^j}\frac{x^p}{\partial \overline{x}^k}\frac{\partial x^q}{\partial \overline{x}^m}g_{qr}$$

$$(3)\frac{\partial \overline{x}_{mj}}{\partial \overline{x}^k} = \frac{\partial x^r}{\partial \overline{x}^m}\frac{\partial x^p}{\partial \overline{x}^j}\frac{\partial x_{rp}}{\partial \overline{x}^q}\frac{\partial x^q}{\partial \overline{x}^k} + \frac{\partial x^r}{\partial \overline{x}^m}\frac{\partial^2}{\partial \overline{x}^k}\frac{x^p}{\partial \overline{x}^j}g_{rp} + \frac{\partial^2}{\partial \overline{x}^k}\frac{x^r}{\partial \overline{x}^m}\frac{\partial x^p}{\partial \overline{x}^j}g_{rp}$$

Subtracting (1) from the sum of (2) and (3) and multiplying by $1/2$, we obtain on using the definition of the Christoffel's symbols of the first kind. Then with appropriate changes of dummy indices we have,

$$(4)\left[\overline{jk,m}\right] = \frac{\partial x^p}{\partial \overline{x}^j}\frac{\partial x^q}{\partial \overline{x}^k}\frac{\partial x^r}{\partial \overline{x}^m}\left[pq,r\right] + \frac{\partial^2}{\partial \overline{x}^j}\frac{x^p}{\partial \overline{x}^k}\frac{\partial x^q}{\partial \overline{x}^m}g_{pq}$$

Where the bar over the Christoffel's symbol indicates that is calculated in the coordinate system $\overline{x}^p$ with respect to its fundamental tensor $\overline{g}_{pq}$.

(b) Multiply (4) by $\overline{g}^{nm} = \frac{\partial \overline{x}^n}{\partial x^s}\frac{\partial \overline{x}^m}{\partial x^t}g^{st}$ to obtain,

$$\overline{g}^{nm}\left[\overline{jk,m}\right]=\frac{\partial x^p}{\partial \overline{x}^j}\frac{\partial x^q}{\partial \overline{x}^k}\frac{\partial x^r}{\partial \overline{x}^m}\frac{\partial \overline{x}^n}{\partial x^s}\frac{\partial \overline{x}^m}{\partial x^t}\left[pq,r\right]$$
$$+\frac{\partial^2}{\partial \overline{x}^j}\frac{x^p}{\partial \overline{x}^k}\frac{\partial x^q}{\partial \overline{x}^m}\frac{\partial \overline{x}^n}{\partial x^s}\frac{\partial \overline{x}^m}{\partial x^t}g^{st}g_{pq}$$

Then

$$\begin{Bmatrix}\overline{n}\\ j\ k\end{Bmatrix}=\frac{\partial x^p}{\partial \overline{x}^j}\frac{\partial x^q}{\partial \overline{x}^k}\frac{\partial \overline{x}^n}{\partial x^s}\delta_t^r g^{st}\left[pq,r\right]$$

$$\begin{Bmatrix}\overline{n}\\ j\ k\end{Bmatrix}=\frac{\partial^2}{\partial \overline{x}^j}\frac{x^p}{\partial \overline{x}^k}\frac{\partial \overline{x}^n}{\partial x^s}\delta_t^q g^{st}g_{pq}$$

$$=\frac{\partial x^p}{\partial \overline{x}^j}\frac{\partial x^q}{\partial \overline{x}^k}\frac{\partial \overline{x}^n}{\partial x^s}\begin{Bmatrix}s\\ p\ q\end{Bmatrix}+\frac{\partial^2}{\partial \overline{x}^j}\frac{x^p}{\partial \overline{x}^k}\frac{\partial \overline{x}^n}{\partial x^p},$$

Since

$$\delta_t^r g^{st}\left[pq,r\right]=g^{sr}\left[pq,r\right]=\begin{Bmatrix}s\\ p\ q\end{Bmatrix}$$

And

$$\delta_t^q g^{st}g_{pq}=g^{sq}g_{pq}=\delta_p^s.$$

Example [2]

Prove

$$\frac{\partial^2}{\partial \overline{x}^j}\frac{x^m}{\partial \overline{x}^k}=\begin{Bmatrix}\overline{n}\\ j\ k\end{Bmatrix}\frac{x^m}{\partial \overline{x}^n}-\frac{x^p}{\partial \overline{x}^j}\frac{x^q}{\partial \overline{x}^k}\begin{Bmatrix}m\\ p\ q\end{Bmatrix}$$

From Example (1) (b),

$$\begin{Bmatrix} \overline{n} \\ j\ k \end{Bmatrix} = \frac{\partial x^p}{\partial \overline{x}^j} \frac{x^q}{\partial \overline{x}^k} \frac{\partial \overline{x}^n}{\partial x^s} \begin{Bmatrix} s \\ p\ q \end{Bmatrix} + \frac{\partial^2}{\partial \overline{x}^j} \frac{x^p}{\partial \overline{x}^k} \frac{\partial \overline{x}^n}{\partial x^p}$$

Multiplying by $\dfrac{\partial x^m}{\partial \overline{x}^n}$,

$$\begin{Bmatrix} \overline{n} \\ j\ k \end{Bmatrix} \frac{\partial x^m}{\partial \overline{x}^n} = \frac{\partial x^p}{\partial \overline{x}^j} \frac{x^q}{\partial \overline{x}^k} \frac{\partial \overline{x}^n}{\partial x^s} \delta_s^m \begin{Bmatrix} s \\ p\ q \end{Bmatrix} + \frac{\partial^2}{\partial \overline{x}^j} \frac{x^p}{\partial \overline{x}^k} \delta_p^m$$

$$= \frac{\partial x^p}{\partial \overline{x}^j} \frac{x^q}{\partial \overline{x}^k} \frac{\partial \overline{x}^n}{\partial x^s} \begin{Bmatrix} s \\ p\ q \end{Bmatrix} + \frac{\partial^2}{\partial \overline{x}^j} \frac{x^m}{\partial \overline{x}^k}$$

Solving for $\dfrac{\partial^2}{\partial \overline{x}^j} \dfrac{x^m}{\partial \overline{x}^k}$, the result follows.

The Covariant Derivative of a Tensor A_p with respect to x^q is denoted by $A_{p,q}$ and is defined by

$$A_{p,q} = \frac{\partial A_p}{\partial x^q} - \begin{Bmatrix} s \\ p\ q \end{Bmatrix} A_s$$

a covariant tensor of rank two.

The covariant derivative of a tensor A^p with respect to x^q is denoted by A^p,q and is defined by

$$A^p,q = \frac{\partial A_p}{\partial x^q} + \begin{Bmatrix} p \\ q\ s \end{Bmatrix} A^s$$

a mixed tensor of rank two.

For rectangular systems, the Christoffel

symbols are zeros and the covariant derivatives are the usual partial derivatives. Covariant derivatives of tensors are also tensors.

The above results can be extended to covariant derivatives of higher rank tensors. Thus,

$$A^{p_1\cdots p_m}_{r_1\cdots r_n},q = \frac{\partial A^{p_1\cdots p_m}_{r_1\cdots r_n}}{\partial x^q} - \begin{Bmatrix} s \\ r_1 \; q \end{Bmatrix} A^{p_1\cdots p_m}_{sr_2\cdots r_n} - \begin{Bmatrix} s \\ r_2 \; q \end{Bmatrix} A^{p_1\cdots p_m}_{r_1 s r_3\cdots r_n} - \cdots$$
$$- \begin{Bmatrix} s \\ r_n \; q \end{Bmatrix} A^{p_1\cdots p_m}_{r_1\cdots r_{n-1}s}$$
$$+ \begin{Bmatrix} p_1 \\ q \; s \end{Bmatrix} A^{sp_2\cdots p_m}_{r_1\cdots r_n} + \begin{Bmatrix} p_2 \\ q \; s \end{Bmatrix} A^{p_1 s p_3\cdots p_m}_{r_1\cdots r_n} + \cdots$$
$$+ \begin{Bmatrix} p_m \\ q \; s \end{Bmatrix} A^{p_1\cdots p_{m-1}s}_{r_1\cdots r_n}$$

is the covariant derivative of $A^{p_1\cdots p_m}_{r_1\cdots r_n}$ with respect to x^q.

The rules of covariant differentiation for sums and products of tensors are the same as those for ordinary differentiation. In performing the differentiation, the tensors g_{pq}, g^{pq} and δ^q_p may be treated as constants since their covariant derivatives are zero.

Since covariant derivatives express rates of change of physical quantities independent of any frame of reference, they are great

importance in expressing physical laws.

Curvature Tensor

Riemann-Christoffel Tensor:

We shall investigates the commutative problem with respect to covariant differentiation. Let us begin with the covariant derivative of an arbitrary covariant vector A_j,

$$A_{j,n} = \frac{\partial A_j}{\partial x^n} - \begin{Bmatrix} l \\ j\ n \end{Bmatrix} A_l \text{..............................[1]}$$

A further covariant differentiation yields,

$$A_{j,np} = \frac{\partial}{\partial x^p}\left(A_{j,n}\right) - \begin{Bmatrix} l \\ j\ p \end{Bmatrix} A_{l,n} - \begin{Bmatrix} l \\ n\ p \end{Bmatrix} A_{j,l}$$

$$= \frac{\partial^2}{\partial x^n}\frac{A_j}{\partial x^p} - \begin{Bmatrix} l \\ j\ n \end{Bmatrix}\frac{\partial A_l}{\partial x^p} - A_l \frac{\partial}{\partial x^p}\begin{Bmatrix} l \\ j\ n \end{Bmatrix} - \begin{Bmatrix} l \\ j\ p \end{Bmatrix}\frac{\partial A_l}{\partial x^n}$$

$$+\begin{Bmatrix} l \\ j\ p \end{Bmatrix}\begin{Bmatrix} k \\ l\ n \end{Bmatrix} A_k - \begin{Bmatrix} l \\ n\ p \end{Bmatrix}\frac{\partial A_j}{\partial x^l}\begin{Bmatrix} l \\ n\ p \end{Bmatrix}\begin{Bmatrix} k \\ j\ l \end{Bmatrix} A_k \text{.....[2]}$$

We interchange the indices n and p and subtract. After changing several dummy indices $(l \to s, k \to l)$ we have,

$$A_{j,np} - A_{j,pn} = \left[\frac{\partial}{\partial x^n} \begin{Bmatrix} l \\ j\ p \end{Bmatrix} - \frac{\partial}{\partial x^p} \begin{Bmatrix} l \\ j\ n \end{Bmatrix} + \begin{Bmatrix} l \\ n\ s \end{Bmatrix} \begin{Bmatrix} s \\ j\ p \end{Bmatrix} - \begin{Bmatrix} l \\ p\ s \end{Bmatrix} \begin{Bmatrix} s \\ j\ n \end{Bmatrix} \right] A_l \quad \ldots\ldots [3]$$

Since A_l is an arbitrary vector, it follows from the quotient law that the expression in square brackets is a mixed tensor of fourth order, of contra-variant order one and covariant order three. Using the notation,

$$R^l_{\bullet jnp} = \frac{\partial}{\partial x^n} \begin{Bmatrix} l \\ j\ p \end{Bmatrix} - \frac{\partial}{\partial x^p} \begin{Bmatrix} l \\ j\ n \end{Bmatrix} + \begin{Bmatrix} l \\ n\ s \end{Bmatrix} \begin{Bmatrix} s \\ j\ p \end{Bmatrix} - \begin{Bmatrix} l \\ p\ s \end{Bmatrix} \begin{Bmatrix} s \\ j\ n \end{Bmatrix} \quad \ldots\ldots [4]$$

we observe that $R^l_{\bullet jnp}$ is a tensor of the fourth order, called the Riemann-Christoffel tensor. This tensor does not depend on the choice of vector A_l. We can now write,

$$A_{j,np} - A_{j,pn} = R^l_{\bullet jnp} A_l \quad \ldots\ldots [$$

It is clear from this equation, that the necessary and sufficient condition that the covariant differentiation of all vectors be commutative, is

that the Riemann-Christoffel tensor be identically zero.

Referring to the definition (4) we observe that,

$$R^{l}_{\bullet j n p} = -R^{l}_{\bullet j p n} \ldots\ldots [*]$$

The in $R^{l}_{\bullet j n p}$ is skew-symmetric with respect to the indices n and p.

Example (1) prove that

$$R^{l}_{\bullet j n p} + R^{l}_{\bullet n p j} + R^{l}_{\bullet p j n} = 0$$

Example (2) prove that $R^{l}_{\bullet l n p} = 0$.

Curvature Tensor

We now introduce the covariant curvature tensor defined by,

$$R_{r j n p} = g_{rl} R^{l}_{\bullet j n p} \ldots\ldots [6]$$

On substituting from (4) into (6) we obtain,

$$R_{r j n p} = \frac{\partial}{\partial x^{n}}\left[g_{rl}\left\{\begin{matrix} l \\ j\ p \end{matrix}\right\}\right] - \frac{\partial g_{rl}}{\partial x^{n}}\left\{\begin{matrix} l \\ j\ p \end{matrix}\right\} - \frac{\partial}{\partial x^{p}}\left[g_{rl}\left\{\begin{matrix} l \\ j\ n \end{matrix}\right\}\right]$$

$$+ \frac{\partial g_{r}}{\partial x^{p}}\left\{\begin{matrix} l \\ j\ n \end{matrix}\right\} + g_{rl}\left\{\begin{matrix} l \\ n\ s \end{matrix}\right\}\left\{\begin{matrix} s \\ j\ p \end{matrix}\right\} - g_{rl}\left\{\begin{matrix} l \\ p\ s \end{matrix}\right\}\left\{\begin{matrix} s \\ j\ n \end{matrix}\right\} \ldots [7]$$

Knowing that,

$$[i\,j,m] = g_{lm}\begin{Bmatrix} l \\ i\ j \end{Bmatrix}$$

$$\frac{\partial g_{ik}}{\partial x^j} = [i\,j,k] + [k\,j,i]$$

and changing the indices $s \to l$

Equation (7) then can be reduced to,

$$R_{r\,j\,n\,p} = \frac{\partial}{\partial x^n}[j\,p,r] - \frac{\partial}{\partial x^p}[j\,n,r] + \begin{Bmatrix} l \\ j\ n \end{Bmatrix}[r\,p,l]$$

$$-\begin{Bmatrix} l \\ j\ p \end{Bmatrix}[r\,n,l] \quad \text{..[8]}$$

Having,

$$[i\,j,k] = \frac{1}{2}\left(\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k}\right),$$

$$\begin{Bmatrix} l \\ i\ j \end{Bmatrix} = g^{lk}[i\,j,k]$$

Equation (8) can be finally reduced to the following important formula,

$$R_{r\,j\,n\,p} = \frac{1}{2}\left(\frac{\partial^2 g_{rp}}{\partial x^j \partial x^n} + \frac{\partial^2 g_{jn}}{\partial x^r \partial x^p} - \frac{\partial^2 g_{rn}}{\partial x^j \partial x^p} - \frac{\partial^2 g_{jp}}{\partial x^r \partial x^n}\right)$$

$$+g\,t\,s\left([j\,n,s][r\,p,t] - [j\,p,s][r\,n,t]\right) \quad \text{..................[9]}$$

From this result we can deduce the relations,

$$\left.\begin{aligned} R_{r\,j\,n\,p} &= -R_{j\,r\,n\,p} \\ R_{r\,j\,n\,p} &= -R_{r\,j\,p\,n} \\ R_{r\,j\,n\,p} &= -R_{n\,p\,r\,j} \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots [10]$$

and,

$$R_{r\,j\,n\,p} + R_{r\,n\,p\,j} + R_{r\,p\,j\,n} = 0 \ldots\ldots\ldots\ldots\ldots\ldots [11]$$

Referring to equation (10) we see that the tensor $R_{r\,j\,n\,p}$ has components equaling zero if either $r = j$ or $n = p$.

From equation (10) we can rewrite equation (11) in the form,

$$R_{j\,r\,n\,p} + R_{j\,p\,r\,n} + R_{j\,n\,p\,r} = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots [12]$$

Contracted curvature tensor, Ricci Tensor:

$$R^{j}_{\bullet pqr} = \begin{Bmatrix} k \\ p\ r \end{Bmatrix}\begin{Bmatrix} j \\ k\ q \end{Bmatrix} - \frac{\partial}{\partial x^{r}}\begin{Bmatrix} j \\ p\ q \end{Bmatrix} - \begin{Bmatrix} k \\ p\ q \end{Bmatrix}\begin{Bmatrix} j \\ k\ r \end{Bmatrix} + \frac{\partial}{\partial x^{q}}\begin{Bmatrix} j \\ p\ r \end{Bmatrix}$$

At first sight there appear to be three different ways of contacting the Riemann-Christoffel tensor $R^{j}_{\bullet pqr}$. We have $R^{j}_{\bullet jqr} = g^{pj} R_{j\,pqr} = 0$ because $R_{j\,pqr}$ is skew –symmetric in $j\,p$. We

see from (*) that $R^{j}_{\bullet pqj} = -R^{j}_{\bullet pjq}$.

Hence we need only consider the contraction, called the Ricci tensor, defined by,

$$R_{pq} = R^{j}_{\bullet pqj} = g^{rj} R_{jpqr}$$

Thus setting $j = r$

$$R^{j}_{\bullet pqj} = \begin{Bmatrix} k \\ p\ j \end{Bmatrix} \begin{Bmatrix} j \\ k\ q \end{Bmatrix} - \frac{\partial}{\partial x^{j}} \begin{Bmatrix} j \\ p\ q \end{Bmatrix} - \begin{Bmatrix} k \\ p\ q \end{Bmatrix} \begin{Bmatrix} j \\ k\ j \end{Bmatrix} + \frac{\partial}{\partial x^{q}} \begin{Bmatrix} j \\ p\ j \end{Bmatrix}$$

Since,

$$\begin{Bmatrix} j \\ j\ m \end{Bmatrix} = \frac{\partial}{\partial x^{m}} \ln \sqrt{g}$$

Therefore we have

$$\begin{aligned} R^{j}_{\bullet pqj} &= \begin{Bmatrix} k \\ p\ j \end{Bmatrix} \begin{Bmatrix} j \\ k\ q \end{Bmatrix} - \frac{\partial}{\partial x^{j}} \begin{Bmatrix} j \\ p\ q \end{Bmatrix} \\ &\quad - \begin{Bmatrix} k \\ p\ q \end{Bmatrix} \frac{\partial}{\partial x^{k}} \ln \sqrt{g} + \frac{\partial^{2}}{\partial x^{p} \partial x^{q}} \ln \sqrt{g} \\ &= R_{pq} \end{aligned}$$

From which it is clear that R_{pq} is symmetric in p, q. (If g is negative, we must replace $\ln \sqrt{g}$ by $\ln \sqrt{-g}$.)

Exercises

1. Write each of the following using the summation convention

a. $\left(x^1\right)^2+\left(x^2\right)^2+\cdots+\left(x^N\right)^2$,

b. $ds^2=g_{11}\left(dx^1\right)^2+g_{22}\left(dx^2\right)^2+g_{33}\left(dx^2\right)^2$,

c. $\sum_{p=1}^{3}\sum_{q=1}^{3}g_{pq}dx^p dx^q$

2. Write the terms in each of the following indicated sums

a. $A_{pq}A^{qr}$,

b. $\overline{g}_{rs}=g_{jk}\dfrac{\partial x^j}{\partial\overline{x}^r}\dfrac{\partial x^k}{\partial\overline{x}^s}$, $N=3$

3. Evaluate (a) $\delta_p^q A_s^{qr}$, (b) $\delta_q^p\delta_r^q$.

4. Show that the velocity of a fluid at any point is a contra-variant tensor of rank one.

5. If A_r^{pq} and B_r^{pq} are tensors, prove that their sum and difference are tensors.

6. Prove that δ_q^p is a mixed tensor of the second rank.

7. If a tensor A_{st}^{pqr} is symmetric (skew-symmetric) with respect to indices p and q in one coordinate system, show that it remains symmetric (skew-symmetric) with

respect to p and q in any coordinate system.

8. Show that every tensor can be expressed as the sum of two tensors, one of which is symmetric and the other skew-symmetric in a pair of covariant or contra-variant indices.

CHAPTER IV
Multiple Integrals

Integral Transformations

Line, Surface, Volume Integrals

1- Line Integrals:

Let $r(u) = x(u)i + y(u)j + z(u)k$, where $r(u)$ is the position vector of (x , y , z), define a curve C joining points P_1 and P_2, where $u = u_1$ and $u = u_2$ respectively.

We assume that C is composed of a finite number of curves for each of which r (u) has a continuous derivative. Let $\underline{A}(x,y,z) = A_1\underline{i} + A_2\underline{j} + A_3\underline{k}$ be a vector function of position defined and continuous along C. Then the integral of the tangential component of $\underline{A}$ along C from P_1 to P_2, written as:

$$\int_{P_1}^{P_2} \underline{A} \cdot \underline{dr} = \int_C \underline{A} \cdot \underline{dr} = \int_C A_1 dx + A_2 dy + A_3 dz$$

is an example of a line integral. If $\underline{A}$ is the force $\underline{F}$ on a particle moving along C, this line integral represents the work done by the force. If C is a closed curve (which we shall suppose is a simple closed curve, i.e. a curve which does not intersect itself anywhere) the integral around C is often denoted by:

$$\oint \underline{A} \cdot \underline{dr} = \oint A_1 dx + A_2 dy + A_3 dz$$

In aerodynamics and fluid mechanics this integral is called the circulation of $\underline{A}$ about C, where $\underline{A}$ represents the velocity of a fluid.

In general, any integral which is to be evaluated along a curve is called a line integral. Such integrals can be defined in terms of limits of sums as are the integrals of elementary calculus.

For methods of evaluation of line integrals, see the Solved. The following theorem is important.

Theorem: If $\underline{A} = \underline{\nabla}\phi$ everywhere in a region R of space, defined by $a_1 \le x \le a_2$, $b_1 \le y \le b_2$, $c_1 \le z \le c_2$ where $\phi(x,y,z)$ is single-valued and has continuous derivatives in R, then:

1- $\int_{P_1}^{P_2} \underline{A}\,.\,\underline{dr}$ is independent of the path C in R joining P_1 and P_2.

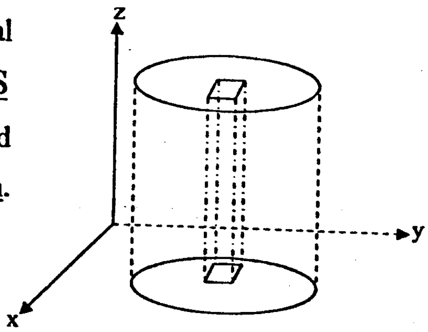
2- $\oint_C \underline{A}\,.\,\underline{dr} = 0$ around any closed curve C in R.

In such case $\underline{A}$ is called a conservative vector field and ϕ is its scalar potential.

A vector field $\underline{A}$ is conservative if and only if $\underline{\nabla} \times \underline{A} = 0$, or equivalently $\underline{A} = \nabla \phi$. In such case $\underline{A} \cdot \underline{dr} = A_1 dx + A_2 dy + A_3 dz = d\phi$, an exact differential.

2- Surface integrals:

Let S be a two-sided surface, such as shown in the figure below. Let one side of S be considered arbitrarily as the positive side(if S is a closed surface this is taken as the outer side). A unit normal $\underline{n}$ to any point of the positive side of S is called a positive or outward drawn unit normal.



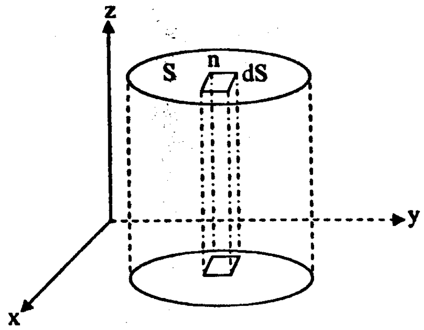
Associate with the differential of surface area dS a vector $\underline{dS}$ whose magnitude is dS and whose direction is that of $\underline{n}$. Then $\underline{dS} = \underline{n}\, dS$.

The integral:

$$\iint_S \underline{A} \cdot \underline{dS} = \iint_S \underline{A} \cdot \underline{n}\, dS$$

Is an example of a surface integral called the flux of $\underline{A}$ over S. Other surface integrals are:

$$\iint_S \phi \, dS \quad , \quad \iint_S \phi \, \underline{n} \, dS \quad , \quad \iint_S \underline{A} \times d\underline{S}$$



where ϕ is a scalar function. Such integrals can be defined in terms of limits of sums as in elementary calculus.

The notation $\oiint_S$ is sometimes used to indicate integration over the closed surface S. Where no confusion can arise the notation $\oint_S$ may be used.

To evaluate surface integrals, it is convenient to express them as double integrals taken over the projected area of the surface 3 on one of the coordinate planes. This is possible if any line perpendicular to the coordinate plane chosen meets the surface in no more than one point.

3- Volume Integrals:

Consider a closed surface in space enclosing a volume **V**. Then:

$$\iiint_V \underline{A}\, dV \qquad \text{and} \qquad \iiint_V \phi\, dV$$

are examples of volume integrals or space integrals as they are sometimes called.

Examples:

Ex.1: If $\underline{A} = (3x^2 + 6y)\underline{i} - 14yz\underline{j} + 20xz^2\underline{k}$, evaluate $\int_C \underline{A}\cdot\underline{dr}$ from (0 , 0 , 0) to (1 , 1 , 1) along the following paths C:

a- $x = t \quad , \quad y = t^2 \quad , \quad z = t^3$

b- The straight line joining (0 , 0 , 0) and (1 , 1 , 1).

$$\int_C \underline{A}\cdot\underline{dr} = \int_C (3x^2 + 6y)\underline{i} - 14yz\underline{j} + 20xz^2\underline{k}\cdot(dx\underline{i} + dy\underline{j} + dz\underline{k})$$

$$= \int_C (3x^2 + 6y)dx - 14yz\, dy + 20xz^2 dz$$

(a) If $x = t$, $y = t^2$, $z = t^3$, points (0 , 0 , 0) and (1 , 1 , 1) correspond to $t = 0$ and $t = 1$ respectively. Then:

$$\int_C \underline{A}\cdot\underline{dr} = \int_{t=0}^{1} (3t^2+6t^2)dt - 14(t^2)(t^3)d(t^2) + 20(t)(t^3)^2 d(t^3)$$

$$= \int_{t=0}^{1} 9t^2dt - 28t^6dt + 60t^9dt$$

$$= \int_{t=0}^{1} (9t^2 - 28t^6 + 60t^9)dt = \left[3t^3 - 4t^7 + 6t^{10}\right]_0^1 = 5$$

Anther Meth d:

Along C , $\underline{A} = 9t^2\underline{i} - 14t^5\underline{j} + 20t^7\underline{k}$ and $\underline{r} = x\underline{i} + y\underline{j} + z\underline{k} = t\underline{i} + t^2\underline{j} + t^3\underline{k}$ and $\underline{dr} = (\underline{i} + 2t\underline{j} + 3t^2\underline{k})dt$.

Then:

$$\int_C \underline{A}\cdot\underline{dr} = \int_{t=0}^{1} (9t^2\underline{i} - 14t^5\underline{j} + 20t^7\underline{k}).(\underline{i} + 2t\underline{j} + 3t^2\underline{k})\, dt$$

$$= \int_0^1 (9t^2 - 28t^8 + 60t^9)\, dt = 5$$

The straight line joining (0 , 0 , 0) and (1 , 1 , 1) is given in parametric form by $x = t$, $y = t$, $z = t$. Then:

$$\int_C \underline{A}\cdot\underline{dr} = \int_{t=0}^{1} (3t^2+6t)dt - 14(t)(t)dt + 20(t)(t)^2 dt$$

$$= \int_{t=0}^{1} (3t^2 + 6t - 14t^2 + 20t^3)dt = (6t - 11t^2 + 20t^3)dt = \frac{13}{3}$$

Ex.2: If $\underline{F} = 3xy\underline{i} - y^2\underline{j}$, evaluate $\int_C \underline{F}\cdot\underline{dr}$ where C is the curve in the x y plane, $y = 2x^2$, from (0 , 0) to (1 , 2).

Since the integration is performed in the x y plane (z = 0), we take $\underline{r} = x\underline{i} + y\underline{j}$. Then:

$$\int_C \underline{F}\cdot\underline{dr} = \int_C \left(3xy\underline{i} - y^2\underline{j}\right).\left(dx\underline{i} + dy\underline{j}\right)$$

$$= \int_C 3xy\,dx - y^2dy$$

First Method: Let x = t in $y = 2x^2$. Then the parametric equations of C are x = t , $y = 2t^2$. Points (0 , 0) and (1 , 2) correspond to t = 0 and t = 1 respectively. Then:

$$\int_C \underline{F}\cdot\underline{dr} = \int_{t=0}^{1} 3(t)\left(2t^2\right)dt - \left(2t^2\right)^2 d\left(2t^2\right)$$

$$= \int_{t=0}^{1} \left(6t^3 - 16t^5\right)dt = -\frac{7}{6}$$

Second Method: Substitute $y = 2x^2$ directly, where x goes from 0 to 1. Then:

$$\int_C \underline{F}\cdot\underline{dr} = \int_{x=0}^{1} 3x\left(2x^2\right)dx - \left(2x^2\right)^2 d\left(2x^2\right)$$

$$= \int_{x=0}^{1} \left(6x^3 - 16x^5\right)dx = -\frac{7}{6}$$

Note that if the curve were traversed in the opposite sense, i.e. from (1 , 2) to (0 , 0), the value of the integral would have been 7/6 instead of -7/6.

Ex.3: If $\phi = 2xyz^2$, $\underline{F} = xy\underline{i} - z\underline{j} + x^2\underline{k}$ and C is the curve $x = t^2$, $y = 2t$, $z = t^3$ from t = 0 to t = 1, evaluate the line integrals:

(a) $\int_C \phi\, \underline{dr}$ (b) $\int_C \underline{F} \wedge \underline{dr}$

(a) Along C, $\phi = 2xyz^2 = 2\left(t^2\right)\left(2t\right)\left(t^3\right)^2 = 4t^9$

$$\underline{r} = x\underline{i} + y\underline{j} + z\underline{k} = t^2\underline{i} + 2t\underline{j} + t^3\underline{k} \text{ and}$$

$$\underline{dr} = \left(2t\underline{i} + 2\underline{j} + 3t^2\underline{k}\right)dt \qquad \text{Then}$$

$$\int_C \phi\, \underline{dr} = \int_{t=0}^{1} 4t^9\left(2t\underline{i} + 2\underline{j} + 3t^2\underline{k}\right)dt$$

$$= \underline{i}\int_0^1 8t^{10}dt + \underline{j}\int_0^1 8t^9 dt + \underline{k}\int_0^1 12t^{11}dt$$

$$= \frac{8}{11}\underline{i} + \frac{4}{5}\underline{j} + \underline{k}$$

(b) Along C, $\underline{F} = xy\underline{i} - z\underline{j} + x^2\underline{k} = 2t^3\underline{i} - t^3\underline{j} + t^4\underline{k}$

Then $\underline{F} \times \underline{dr} = \left(2t^3\underline{i} - t^3\underline{j} + t^4\underline{k}\right) \times \left(2t\underline{i} + 2\underline{j} + 3t^2\underline{k}\right)dt$

$$= \begin{vmatrix} i & j & k \\ 2t^3 & -t^3 & t^4 \\ 2t & 2 & 3t^2 \end{vmatrix} dt$$

$$= \left[\left(-3t^5 - 2t^4\right)\underline{i} + \left(2t^5 - 6t^5\right)\underline{j} + \left(4t^3 + 2t^4\right)\underline{k}\right]dt$$

and $$\int_C \underline{F} \times \underline{dr} = \underline{i}\int_0^1 \left(-3t^5 - 2t^4\right)dt$$

$$+\underline{j}\int_0^1 \left(-4t^5\right)dt + \underline{k}\int_0^1 \left(4t^3 + 2t^4\right)dt$$

$$= \frac{9}{10}\underline{i} - \frac{2}{3}\underline{j} + \frac{7}{5}\underline{k}$$

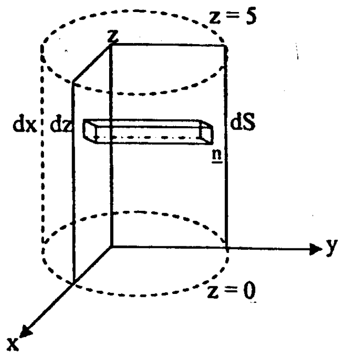
Ex.4: Evaluate $\int_S \underline{A} \cdot \underline{n}\ dS$, where $\underline{A} = z\underline{i} + x\underline{j} - 3y^2z\underline{k}$ and S is the surface of the cylinder $x^2 + y^2 = 16$ included in the first octant between $z = 0$ and $z = 5$.

Project S on the x z plane as in the figure below and call the projection R. Note that the projection of S on the x y plane can note be used here. Then:

$$\iint_S \underline{A} \cdot \underline{n} \, dS = \iint_R \underline{A} \cdot \underline{n} \frac{dx\,dz}{|\underline{n} \cdot \underline{j}|}$$

A normal to $x^2 + y^2 = 16$ is $\nabla(x^2 + y^2) = 2x\underline{i} + 2y\underline{j}$

Thus the unit normal to S as shown in the adjoining figure, is:



$$\underline{n} = \frac{2x\underline{i} + 2y\underline{j}}{\sqrt{(2x)^2 + (2y)^2}} = \frac{x\underline{i} + y\underline{j}}{4}$$

since $x^2 + y^2 = 16$ on S.

$$\underline{A} \cdot \underline{n} = (z\underline{i} + x\underline{j} - 3y^2 z\underline{k}) \cdot \left(\frac{x\underline{i} + y\underline{j}}{4}\right) = \frac{1}{4}(xz + xy)$$

$$\therefore \quad \underline{n} \cdot \underline{j} = \frac{x\underline{i} + y\underline{j}}{4} \cdot \underline{j} = \frac{y}{4}$$

Then the surface integral equals:

$$\iint_R \frac{xz + xy}{y} dx\,dz = \int_{z=0}^{5} \int_{x=0}^{4} \left(\frac{xz}{\sqrt{16 - x^2}} + x\right) dx\,dz$$

$$= \int_{z=0}^{5} (4z + 8)\,dz = 90$$

Reduction of Volume to Surface Integrals:

4- Gauss's Divergence Theorem:

If F is a continuously differentiable vector point function and S is a closed surface enclosing a region V, Then:

$$\int_S \underline{F}\cdot\underline{n}\ dS = \int_V \operatorname{div}\underline{F}\ dV \qquad (1)$$

where $\underline{n}$ is the unit outward drawn normal vector.

Briefly, the theorem states that the normal surface integral of a function $\underline{F}$ over the boundary of a closed region is equal to the volume integral of div $\underline{F}$ taken throughout the region.

To start with, we shall suppose that the region V is such that it is possible to choose a rectangular cartesian co-ordinate system such that each line parallel to any co-ordinate axis which has internal points in common with the region meets the boundary S in two points.

In terms of cartesian co-ordinates, the equation (1) is stated as follows:

$$\int_S (F_1\, dy\, dz + F_2\, dz\, dx + F_3\, dx\, dy)$$

$$= \iiint_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dx\, dy\, dz$$

Consider now the volume integral:

$$\int_V \frac{\partial F_3}{\partial z} dx\, dy$$

Let R_3, denote the projection of the region V, on the OXY plane Every line through a point (x , y , 0) of R_3, meets the boundary S in two points. Let the z, co-ordinates of these points be:

$$z = \phi(x,y) \quad , \quad z = \psi(x,y)$$

Where:

$$\phi(x,y) \geq \psi(x,y)$$

Now:

$$\iiint_V \frac{\partial F_3}{\partial z} dx\, dy\, dz = \iint_{R_3} \left[\int_\psi^\phi \frac{\partial F_3}{\partial z} dz \right] dx\, dy$$

$$= \iint_{R_3} \left[F_3(x,y,\phi) - F_3(x,y,\psi) \right] dx\, dy$$

$$= \iint_{R_3} F_3(x,y,\phi) dx\, dy - \iint_{R_3} F_3(x,y,\psi) dx\, dy \qquad (1)$$

Let us now denote the parts of the surface S corresponding to $z = \phi(x,y)$ and $z = \psi(x,y)$ by S_1 and S_2 respectively.

If $\underline{n}$ denotes the outward drawn unit normal vector at any point of S, we have:

$$\iint_{R_3} F_3(x,y,\phi)\,dx\,dy = \iint_{S_1} F_3\,\underline{n}\,.\,\underline{k}\,dS \qquad (2)$$

$$\iint_{R_3} F_3(x,y,\psi)\,dx\,dy = \iint_{S_2} F_3\,\underline{n}\,.\,\underline{k}\,dS \qquad (3)$$

for the outward drawn normal at any point of S_1, makes an acute angle with positive direction of z-axis and that at any point of S_2 makes an obtuse angle with z-axis.

From (1), (2) and (3), we have:

$$\iiint_V \frac{\partial F_3}{\partial z}\,dx\,dy\,dz = \iint_{S_1} F_3\,\underline{n}\,.\,\underline{k}\,dS + \iint_{S_2} F_3\,\underline{n}\,.\,\underline{k}\,dS$$

$$= \iint_S F_3\,\underline{n}\,.\,\underline{k}\,dS \qquad (4)$$

Similarly it may be proved that:

$$\iiint_V \frac{\partial F_2}{\partial y}\,dx\,dy\,dz = \iint_S F_2\,\underline{n}\,.\,\underline{j}\,dS \qquad (5)$$

$$\iiint_V \frac{\partial F_1}{\partial x}\,dx\,dy\,dz = \iint_S F_1\,\underline{n}\,.\,\underline{i}\,dS \qquad (6)$$

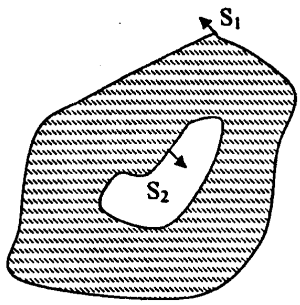
Adding (4) , (5) and (6), we get:

$$\iiint_V \left(\frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}+\frac{\partial F_3}{\partial z}\right) dx\,dy\,dz$$

$$= \iint_S \left(F_1\,\underline{i}+F_2\,\underline{j}+F_3\,\underline{k}\right).\underline{n}\,dS$$

i.e. $$\int_V \operatorname{div}\underline{F}\,dV = \int_S \underline{F}.\underline{n}\,dS = \int_S \underline{F}.\underline{da} \qquad (7)$$

Note: The theorem also holds good for a region V(shaded) such as that shown in the figure. It is bounded, by two closed surfaces S_1 and S_2 one of which lies within the other. It should.



However, be understood that the outward drawn normal at any point means the normal directed away from the region- Thus the normals at points of S_1 and S_2 are to be directed as shown in the figure.

The theorem can also be applied to regions enclosed by everal surfaces.

Cor.1: The Gauss's theorem obtained above enables us to express a normal surface integral as a volume integral. We shall now see that with the help of this theorem we are able to express the other two types of surface integrals also as volume integrals.

It will be shown that:

(i) $$\int_S \underline{F} \wedge \underline{n} \; dS = \int_S \underline{F} \wedge \underline{da} = -\int_V \text{curl}\, \underline{F} \; dV$$

(ii) $$\int_S \phi \, \underline{n} \; dS = \int_S \phi \, \underline{da} = \int_V \underline{\nabla} \phi \; dV$$

We write:

$$\underline{f} = \underline{a} \wedge \underline{F}$$

where $\underline{a}$ is any constant vector and apply Gauss's theorem to the vector function $\underline{f}$. Thus we have:

$$\int_S \underline{a} \wedge \underline{F} \, . \, \underline{n} \, dS = \int_V \text{div}(\underline{a} \wedge \underline{F}) \; dV \qquad (1)$$

But:

$$\underline{a} \wedge \underline{F} \, . \, \underline{n} = \underline{a} \, . \, \underline{F} \wedge \underline{n} \qquad (2)$$

and $$\text{div}(\underline{a} \wedge \underline{F}) = \underline{\nabla} \cdot (\underline{a} \, . \, \underline{F}) = -\underline{a} \, . \, \underline{\nabla} \wedge \underline{F} \qquad (3)$$

From (1), (2) and (3):

$$\int_S \underline{a}.\underline{F}\wedge\underline{n}\ dS = -\int_V \underline{a}.\underline{\nabla}\wedge\underline{F}\ dV$$

or $$\mathbf{a}.\int_S \underline{F}\wedge\underline{n}\ dS = -\underline{a}.\int_V \underline{\nabla}\wedge\underline{F}\ dV$$

or $$\mathbf{a}.\left[\int_S \underline{F}\wedge\underline{n}\ dS + \int_V \underline{\nabla}\wedge\underline{F}\ dV\right] = 0 \qquad (4)$$

As, however, $\underline{a}$ is any arbitrary vector, we have from (4):

$$\int_S \underline{F}\wedge\underline{n}\ dS + \int_V \underline{\nabla}\wedge\underline{F}\ dV = 0$$

i.e. $$\int_S \underline{F}\wedge\underline{n}\ dS = -\int_V \underline{\nabla}\wedge\underline{F}\ dV = -\int_V \mathrm{curl}\,\underline{F}\ dV$$

Thus we have proved (i).

To prove (ii) we write:

$$\underline{f} = \underline{a}\,\phi$$

where $\underline{a}$ is a constant vector. Applying Gauss's theorem to the function f, we have:

$$\int_S \underline{f}.\underline{n}\,dS = \int_V \mathrm{div}\,\underline{f}\ dV$$

i.e. $$\int_S \underline{a}\,\phi\,.\,\underline{n}\,dS = \int_V \mathrm{div}(\underline{a}\,\phi)\;dV = \int_V \underline{\nabla}\cdot(\underline{a}\,\phi)\,dV$$

or $$\mathbf{a}\,.\int_S \phi\,\underline{n}\;dS = \underline{a}\,.\int_V \underline{\nabla}\phi\;dV$$

or $$\mathbf{a}\,.\left[\int_S \phi\,\underline{n}\;dS - \int_V \underline{\nabla}\phi\;dV\right] = 0$$

The vector $\underline{a}$ being arbitrary, we obtain:

$$\int_S \phi\,\underline{n}\,dS - \int_V \underline{\nabla}\phi\;dV = 0$$

or $$\int_S \phi\,\underline{n}\,dS = \int_V \underline{\nabla}\phi\;dV$$

which is (ii).

Note: The three results obtained above can be re-written as follows:

$$\int_S \underline{n}\,.\,\underline{F}\,dS = \int_V \underline{\nabla}\,.\,\underline{F}\;dV$$

$$\int_S \underline{n}\wedge\underline{F}\,dS = \int_V \underline{\nabla}\wedge\underline{F}\;dV$$

$$\int_S \underline{n}\,\phi\;dS = \int_V \underline{\nabla}\phi\;dV$$

It will thus be seen that each of these results can be written by just changing $\underline{n}$ to $\underline{\nabla}$.

The integrand in each case has been so written that n appears before the function and not after.

5- Physical Interpretation of Gauss's theorem:

Let the vector point function $\underline{q}$, denote the velocity vector of an incompressible fluid of unit density and let S denote any closed surface drawn in the fluid. By Gauss's Theorem:

$$\int_S \underline{q}\cdot\underline{n}\, dS = \int_V \operatorname{div}\underline{q}\ dV$$

Now $\underline{q}\cdot\underline{n}$ is the component of the velocity at any point of S in the direction of the outward drawn normal so that $\underline{q}\cdot\underline{n}\,\delta S$ denotes the amount of fluid that flows out in unit time through the element δS. Hence the left hand side of (I) denotes the amount of fluid flowing across the surface S in unit time from the inside to the outside. This amount may be positive, negative or zero.

Now the total amount flowing outwards must be continuously supplied so that inside the region we must have sources producing fluid. We have already seen that div $\underline{q}$ at any

point denotes the amount of fluid per unit time per unit volume that goes through any point. Thus div $\underline{q}$ may be thought of as the source-intensity of the incompressible fluid at any point P.

Hence the right hand side of (I) denotes the amount of fluid per unit time supplied by the sources within S. Thus the equality (1) appears intuitively evident.

Examples:

Ex.1: Show that:

$$\frac{1}{3}\int_S \underline{r}\,.\,d\underline{a} = V$$

where V is the volume enclosed by the surface S. We have:

$$\int_S \frac{1}{3}\underline{r}\,.\,d\underline{a} = \int_V \operatorname{div}\frac{1}{3}\underline{r}\,dV$$

$$= \int_V dV = V, \text{for} \quad \operatorname{div}\underline{r} = 3$$

Ex.2: If $OA = a\underline{i}$, $OB = a\underline{j}$, $OC = a\underline{k}$, form three coterminous edges of a cube and S denotes the surface of the cube, evaluate:

$$\int_S \left[\left(x^3 - yz\right)\underline{i} - 2x^2y\underline{j} + 2\underline{k}\right].\underline{a}\ dS$$

by expressing it as a volume integral. Also verify the result by direct evaluation of the surface integral.

Let:

$$\underline{F} = (x^2 - yz)\underline{i} - 2x^2y\underline{j} + 2\underline{k}$$

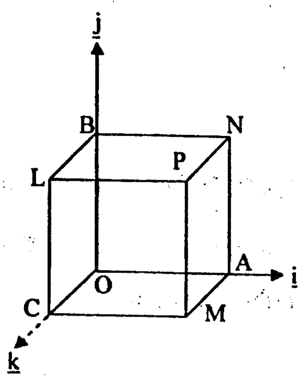
$$\text{div}\,\underline{F} = \frac{\partial(x^2 - yz)}{\partial x} + \frac{\partial(-2x^2y)}{\partial y} + \frac{\partial(2)}{\partial z}$$

$$= 3x^2 - 2x^2 = x^2$$

$$\int_S \underline{F}.\underline{n}\ dS = \int_V \text{div}\,\underline{F}\ dV = \iiint_V x^2 dx\ dy\ dz$$

$$= \int_0^a dz \int_0^a dy \int_0^a x^2 dx$$

$$= a.a.\frac{1}{3}a^3 = \frac{1}{3}a^5$$



We shall now evaluate the surface integral directly. The surface S consists of six faces. Over the surfaces AOBN:

$$\int_S \underline{F}.\underline{n}\ dS$$

$$= \int_S \left[(x^3 - yz)\underline{i} - 2x^2y\underline{j} + 2\underline{k} \right] . (-\underline{k}) dS$$

$$= \int_0^a \int_0^a -dx\, dy = -2a^2$$

Over the surface PLCM:

$$\int_S \underline{F} . \underline{n} \; dS = \int_S (x^3 - yz)\underline{i} - 2x^2y\underline{j} + 2\underline{k} . \underline{k} \, dS$$

$$= \int_0^a \int_0^a 2dx\, dy = 2a^2$$

Similarly over the faces NPMA , BLCO , AOCM , NBLP , the corresponding surface integrals are respectively:

$$\int_S \underline{F} . \underline{i} \; dS = \iint_S (x^3 - yz) dy\, dz = \int_0^a \int_0^a (x^3 - yz) dy\, dz$$

$$= a^5 - \tfrac{1}{4} a^4$$

$$\int_S \underline{F} . (-\underline{i}) \; dS = -\iint_S (x^3 - yz) dy\, dz = \int_0^a \int_0^a yz\, dy\, dz = \tfrac{1}{4} a^4$$

$$\int_S \underline{F} . (-\underline{j}) \; dS = \int_0^a \int_0^a 2x^2y\, dx\, dz = 0$$

$$\int_S \underline{F}.\underline{j}\ dS = -\int_0^a \int_0^a 2x^2 y\ dx\ dz = -2a \int_0^a \int_0^a x^2\ dx\ dz$$

$$= -\frac{2}{3} a^5$$

Adding we see that over the whole surface:

$$\int_S \underline{F}.\underline{n}\ dS = 2a^2 + 2a^2 + a^5 = -\frac{1}{4} a^4 + \frac{1}{4} a^4 + 0 - \frac{2}{3} a^5 = \frac{1}{3} a^5$$

Hence the verification.

Ex.3: Convert:

$$\int_S \underline{F} \wedge d\underline{a}$$

taken over a closed surface into a volume integral and verify the result in the case:

$$F = x\,y\underline{k}$$

where S is the surface of a cube whose base is in the XY plane and which is bounded by the planes:

$$x = 0 \quad , \quad x = a \quad ; \quad y = 0 \quad , \quad y = a$$

Solution:

In Cor.1 to 4, it has been shown that:

$$\int_S \underline{F} \wedge d\underline{a} = -\int_S \text{curl}\, \underline{F}\, dV$$

Now when $\underline{F} = x\,y\underline{k}$, we have:

$$\text{curl}\, \underline{F} = \underline{i}x - \underline{j}y$$

Now
$$\int_V \text{curl}\, \underline{F}\, dV = \underline{i}\int_V x\, dV - \underline{j}\int_V y\, dV$$

$$= \underline{i}\int_0^a \int_0^a \int_0^a x\ dx\, dy\, dz = \underline{j}\int_0^a \int_0^a \int_0^a y\ dx\, dy\, dz$$

$$= \underline{i}\frac{a^4}{2} - \underline{j}\frac{a^4}{2} = \frac{1}{2}a^4\left(\underline{i} - \underline{j}\right)$$

we shall now calculate:

$$\int_S \underline{F} \wedge d\underline{a}$$

Over the six faces of the cube.

Over the face AOCM:

$$\int \underline{F} \wedge d\underline{a} = \iint \left(x\,y\underline{k}\right) \wedge -\underline{j}\, dx\, dz = \underline{i}\int_0^a \int_0^a xy\ dx\, dz = 0$$

for y is zero at every pint of this face.

Over the face PNBL:

$$\int \underline{F} \wedge d\underline{a} = \iint (xy\underline{k}) \wedge \underline{j}\, dx\, dz = -\underline{i}\int_0^a \int_0^a xy\, dx\, dz$$

$$= -\underline{i}a\int_0^a \int_0^a x\, dx\, dz = -\underline{i}\frac{a^4}{2}$$

Similarly, over the face NPMA, BLCO, AOBN , MCLP, the corresponding surface integrals respectively are:

$$\int \underline{F} \wedge d\underline{a} = \iint xy\underline{k} \wedge \underline{i}\, dy\, dz = \underline{j}a\int_0^a \int_0^a y\, dy\, dz = \underline{j}\frac{a^4}{2}$$

$$\int \underline{F} \wedge d\underline{a} = \iint xy\underline{k} \wedge (-\underline{i}\, dy\, dz) = 0$$

for $F = 0$ at every point.

$$\int \underline{F} \wedge d\underline{a} = \iint xy\underline{k} \wedge -\underline{k}\, dx\, dy = 0$$

$$\int \underline{F} \wedge d\underline{a} = \iint xy\underline{k} \wedge \underline{k}\, dx\, dy = 0$$

over the six faces.

$$\int \underline{F} \wedge d\underline{a} = 0 - \underline{i}\frac{a^4}{2} + \underline{j}\frac{a^4}{2} + 0 + 0 + 0 = \frac{1}{2}a^4(\underline{i} - \underline{j})$$

Reduction of surf ace to Line Integrals:

6- Stoke's Theorem:

If $\underline{F}$ is any continuously differentiable vector point func.

and S is a surface bounded by a curve C, then:

$$\int_C \underline{F}\,.\,d\underline{r} = \int_S \text{curl}\,\underline{F}\,.\,\underline{n}\ \ dS$$

where the unit normal vector n at any point of S is drawn in the sense in which a right handed screus would move when rotated in the sense of description of C.

Stoke's Theorem for plane. Firstly we shall prove the theorem for the case of a plane surface. Stoke's theorem in a plan is also often referred to as Green's theorem. Take a system of cartesian rectangular co-ordinate axes such that the plane of the given surface S is the OXY plane and the z-axis lies along the direction of the normal vector n. In the present case of plane the normal vector is constant.

Let:

$$\underline{F} = \underline{i}F_1 + \underline{j}F_2 + \underline{k}F_3$$

We have:

$$\int_C \underline{F}.d\underline{r} = \int_C \underline{F}.\underline{t}\ dS$$

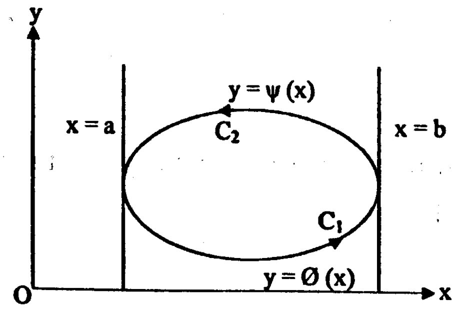
for $dz / dx = 0$; the tangent at any point lying in the OXY plane.

Also $$\int_S \text{curl}\underline{F}.\underline{n}\ dS = \int_C \text{curl}\underline{F}.\underline{k}\ dS$$

$$= \int_S \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\, dy$$

Thus for the case of a plane surface the theorem is equivalent to showing:

$$\int_C (F_1 dx + F_2 dy) = \iint_S \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\, dy$$



We shall now prove it. Firstly suppose that the region S is such that any line parallel to either axis C in at the most two points. Let the region meets be included between the lines x = a

, x = b and any line parallel to y-axis with abscissa J x meet C in the points give by:

$$y = \psi(x) \quad , \quad y = \phi(x)$$

where $\psi(x) \geq \phi(x)$ Thus the boundary curve G is split up into two arcs C_1, and C_2 as shown in the figure.

We have:

$$\iint_S \frac{\partial F_1}{\partial y} dx\, dy = \int_a^b \int_{\phi(x)}^{\psi(x)} \left[\frac{\partial F_1}{\partial y} dy \right] dx$$

$$= \int_a^b F_1[x, \psi(x)] - F_1[x, \phi(x)] dx$$

$$= -\int_{C_2} F_1(x,y) dx - \int_{C_1} F_1(x,y) dx$$

$$= -\int_C F_1(x,y) dx \qquad (1)$$

Similarly we may show that:

$$\iint_S \frac{\partial F_2}{\partial x} dx\, dy = \int_C F_2(x,y) dy \qquad (2)$$

From (1) and (2), we have:

$$\int_C \left(F_2 dy + F_1 dx\right) = \iint_S \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\, dy$$

Now if a plane region be such that it can be divided into a finite number of sub-regions such that the boundary of each is met in at the most two points by any line parallel to either is met in at the most two points by any line parallel to either axis, then we can see the truth of the theorem for the region by applying the result obtained to each sub-region and adding the results. This is because the line integrals along boundary curves will cancel in pairs.

Stoke's Theorem in Space. Suppose the surface of integration S is given by:

$$\underline{r} = \underline{f}(u, v)$$

We represent u , v by a point in a plane, taking the same as the cartesian rectangular co-ordinates of the point. Then the surface S arises as the image of a certain region D of the (u , v) plane such that to each point (u , v) of D, there corresponds one and only one point of S. Let K be the boundary of D whose sense of description corresponds to that of the curve C bounding the surface S.

We have:

$$\int_S \text{curl}\, \underline{F} \,.\, \underline{n}\, dS = \iint_S \left(\frac{\partial F_2}{\partial y} - \frac{\partial F_2}{\partial z}\right) dy\, dz$$

$$+\left(\frac{\partial F_1}{\partial z} - \frac{\partial F_2}{\partial x}\right) dz\, dx + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\, dy$$

$$= \iint_D \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\frac{\partial(y,z)}{\partial(u,v)} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_2}{\partial x}\right)\frac{\partial(z,x)}{\partial(u,v)}$$

$$+\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\frac{\partial(x,y)}{\partial(u,v)} du\, dv$$

In the integrand corresponding to the double integral over the domain D, in the u , v plane, the terms involving F_1 are:

$$\frac{\partial F_1}{\partial z}\frac{\partial(z,x)}{\partial(u,v)} - \frac{\partial F_1}{\partial y}\frac{\partial(x,y)}{\partial(u,v)}$$

$$= -\frac{\partial F_1}{\partial z}\frac{\partial(x,z)}{\partial(u,v)} - \frac{\partial F_1}{\partial y}\frac{\partial(x,y)}{\partial(u,v)} - \frac{\partial F_1}{\partial x}\frac{\partial(x,x)}{\partial(u,v)}$$

the last term introduced being zero. Again these terms are:

$$= -\frac{\partial F_1}{\partial z}\left[\frac{\partial x}{\partial u}\frac{\partial z}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial z}{\partial u}\right] - \frac{\partial F_1}{\partial y}\left[\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right]$$

$$-\frac{\partial F_1}{\partial x}\left[\frac{\partial x}{\partial u}\frac{\partial x}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial x}{\partial u}\right]$$

$$= -\frac{\partial x}{\partial u}\left[\frac{\partial F_1}{\partial z}\frac{\partial z}{\partial v} + \frac{\partial F_1}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial F_1}{\partial x}\frac{\partial x}{\partial v}\right]$$

$$+\frac{\partial x}{\partial v}\left[\frac{\partial F_1}{\partial z}\frac{\partial z}{\partial u} + \frac{\partial F_1}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial F_1}{\partial x}\frac{\partial x}{\partial u}\right]$$

$$= -\frac{\partial x}{\partial u}\frac{\partial F_1}{\partial v} + \frac{\partial x}{\partial v}\frac{\partial F_1}{\partial u} = \frac{\partial F_1}{\partial u}\frac{\partial x}{\partial u} - \frac{\partial F_1}{\partial v}\frac{\partial x}{\partial u}$$

Similarly the terms involving F_2 , F_3 are:

$$\frac{\partial F_2}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial F_2}{\partial v}\frac{\partial y}{\partial u},\ \frac{\partial F_3}{\partial u}\frac{\partial z}{\partial v} - \frac{\partial F_3}{\partial v}\frac{\partial z}{\partial u}$$

Thus the double integral over the domain in the (u , v) plans is:

$$\iint_D \left[\left(\frac{\partial F_1}{\partial u}\frac{\partial x}{\partial v} - \frac{\partial F_1}{\partial v}\frac{\partial x}{\partial u}\right) + \left(\frac{\partial F_2}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial F_2}{\partial v}\frac{\partial y}{\partial u}\right)\right.$$

$$\left. + \left(\frac{\partial F_3}{\partial u}\frac{\partial z}{\partial v} - \frac{\partial F_3}{\partial v}\frac{\partial z}{\partial u}\right)\right] du\, dv$$

By Stoke's theorem in plane:

$$\iint_D \left(\frac{\partial F_1}{\partial u}\frac{\partial x}{\partial v} - \frac{\partial F_1}{\partial v}\frac{\partial x}{\partial u}\right) du\, dv$$

$$= \int_C \left(F_1 \frac{\partial x}{\partial u} du + F_1 \frac{\partial x}{\partial v} dv\right) du\, dv$$

$$= \int_C F_1\, dx$$

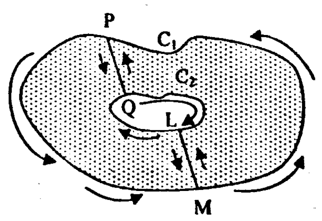
and similarly for the other two parts of the double integral over D. Thus:

$$\int_S \text{curl}\,\underline{F}\,.\,\underline{n}\; dS = \int_C (F_1 dx + F_2 dy + F_3 dz)$$

$$\int_S \text{curl}\,\underline{F}\,.\,\underline{n}\; dS = \int_C \underline{F}\,.\,d\underline{r}$$

Note: The theorem also holds good for a surface S (shaded) such as that shown in the figure. It is enclosed by two curves C_1 and C_2. we have:

$$\int_S \text{curl}\,\underline{F}\,.\,\underline{n}\; dS = \int_{C_1} \underline{F}\,.\,d\underline{r} + \int_{C_2} \underline{F}\,.\,d\underline{r}$$



where the line integrals on the right are taken along the directions of C_1 and C_2 as shown in the figure. The result will follow, if we join the curves by any lines PQ and LM as shown and apply the theorem to the two surfaces thus obtained. The

line integrals along PQ and LM will be taken in opposite directions and will consequently vanish on addition.

Cor.1: We write:

$$\underline{F} = \underline{a}\,\phi$$

where $\underline{a}$ is a constant vector and ϕ, a continuously differentiable scalar point function. Applying Stoke's theorem, we have:

$$\int_C \underline{a}\,\phi\,.\,d\underline{r} = \int_S \operatorname{curl}(\underline{a}\,\phi)\,.\,\underline{n}\;dS \qquad (1)$$

Also:

$$\operatorname{curl}(\underline{a}\,\phi) = \underline{\nabla} \wedge (\underline{a}\,\phi)$$

$$= \underline{\nabla}\phi \wedge \underline{a} \qquad (2)$$

From (1) and (2):

$$\underline{a}\,.\int_C \phi\;dr = \int_S \underline{\nabla}\,\phi \wedge \underline{a}\,.\,\underline{n}\;dS$$

$$= -\underline{a}\,.\int_S \underline{\nabla}\,\phi \wedge \underline{n}\;dS$$

or $$\underline{a}\,.\left[\int_C \phi\ dr + \int_S \underline{\nabla}\,\phi \wedge \underline{n}\ dS\right] = 0$$

$$\therefore \int_C \phi\ dr + \int_S \underline{\nabla}\,\phi \wedge \underline{n}\ dS = 0$$

$\underline{a}$ being any arbitrary vector.

$$\therefore \int_C \phi\ dr = \int_S \underline{n} \wedge \underline{\nabla}\,\phi\ dS$$

2- Again, we write:

$$\underline{f} = \underline{a} \wedge \underline{F}$$

where $\underline{a}$ is a constant vector and apply Stoke's theorem, to the function $\underline{f}$.

$$\therefore \int_C \underline{a} \wedge \underline{F}\,.\,dr = \int_S \left[\text{curl}\,(\underline{a} \wedge \underline{F})\right].\,\underline{n}\ dS$$

Also: $$\text{curl}\,(\underline{a} \wedge \underline{F}) = \underline{\nabla}(\underline{a} \wedge \underline{F}) = \underline{a}(\underline{\nabla}\,.\,\underline{F}) - (\underline{a}\,.\,\underline{\nabla})\underline{F}$$

And: $$\left[(\underline{a}\,.\,\underline{\nabla})\underline{F}\right].\,\underline{n} = \underline{a}\,.\,\underline{\nabla}(\underline{F}\,.\,\underline{n})$$

Where it is understood that ∇ does not operate on $\underline{n}$ and for the purpose of ∇ , $\underline{n}$ is thought of as constant. Thus:

$$\int_C \underline{a}\,.\,\underline{F} \wedge d\underline{r} = \int_S \underline{a}\,.\,(\underline{\nabla}\,.\,\underline{F})\underline{n} - \underline{a}\,.\,\underline{\nabla}(\underline{F}\,.\,\underline{n})\ dS$$

or $$\underline{a}\cdot\int_C \underline{F}\wedge d\underline{r}=\underline{a}\cdot\int_S (\underline{\nabla}\cdot\underline{F})\underline{n}-\underline{\nabla}(\underline{F}\cdot\underline{n})\,dS$$

so that, as before a being a constant vector:

$$\int_C \underline{F}\wedge d\underline{r}=\int_S (\underline{\nabla}\cdot\underline{F})\underline{n}-\underline{\nabla}(\underline{F}\cdot\underline{n})\,dS$$

Looking upon ∇ as a vector, we may write:

$$(\underline{\nabla}\cdot\underline{F})\underline{n}-\underline{\nabla}(\underline{F}\cdot\underline{n})=-(\underline{n}\wedge\underline{\nabla})\wedge\underline{F}$$

Hence: $$\int_C d\underline{r}\wedge\underline{F}=\int_S (\underline{n}\wedge\underline{\nabla})\wedge\underline{F}\,dS$$

The three results connecting line and surface integrals which we have arrived at may now be restated as follows:

$$\int_C d\underline{r}\cdot\underline{F}=\int_S (\underline{n}\wedge\underline{\nabla})\cdot\underline{F}\,dS$$

$$\int_C d\underline{r}\wedge\underline{F}=\int_S (\underline{n}\wedge\underline{\nabla})\wedge\underline{F}\,dS$$

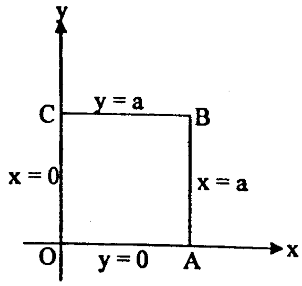
$$\int_C d\underline{r}\;\phi=\int_S (\underline{n}\wedge\underline{\nabla})\phi\,dS$$

so that we have replaced $d\underline{r}$ by $\underline{n}\wedge\underline{\nabla}$. It should also be understood that ∇ on the right does not operate on $\underline{n}$.

Ex.1: Verify Stoke's theorem for the function:

$$\underline{F} = x(\underline{i}x + \underline{j}y)$$

integrated round the square in the plane z = 0 whose sides are along the lines. We have:



$$\text{curl}\, x(\underline{i}x + \underline{j}y) = \underline{k}y$$

$$\therefore \int_S \text{curl}\, x(\underline{i}x + \underline{j}y) . d\underline{a}$$

$$= \int_0^a \int_0^a \underline{k}y . \underline{k}dx\, dy =$$

$$= \int_0^a \int_0^a y\, dx\, dy = \frac{a^3}{2}$$

Again:

$$\int_C \underline{F}.d\underline{r} = \int_{OA} \underline{F}.d\underline{r} + \int_{AB} \underline{F}.d\underline{r} + \int_{BC} \underline{F}.d\underline{r} + \int_{CO} \underline{F}.d\underline{r}$$

$$\int_{OA} \underline{F}.d\underline{r} = \int_0^a x(\underline{i}x + \underline{j}y).\underline{i}\ dx = \int_0^a x^2 dx = \frac{1}{2}a^3$$

$$\int_{AB} \underline{F}.d\underline{r} = \int_0^a x(\underline{i}x + \underline{j}y).\underline{j}\ dy = \int_0^a ay\ dy = \frac{1}{2}a^3$$

$$\int_{BC} \underline{F}.d\underline{r} = \int_0^a x(\underline{i}x + \underline{j}y).\underline{i}\ dx = -\int_0^a x^2 dx = -\frac{1}{2}a^3$$

$$\int_{CO} \underline{F}.d\underline{r} = \int_0^a x(\underline{i}x + \underline{j}y).\underline{j}\ dy = 0$$

$$\therefore \int_C \underline{F}.d\underline{r} = \frac{1}{2}a^2 + \frac{1}{2}a^2 - \frac{1}{2}a^2 + 0 = \frac{1}{2}a^3$$

Hence the verification.

Ex.2: Find the value of:

$$\int \text{curl}\ \underline{F}.d\underline{a}$$

taken over the portion of the surface:

$$x^2 + y^2 - 2ax + az = 0$$

for which $z \geq 0$, when:

$$\underline{F} = (x^2 + z^2 - x^2)\underline{i} + (z^2 + x^2 - y^2)\underline{j} + (x^2 + y^2 - z^2)\underline{k}$$

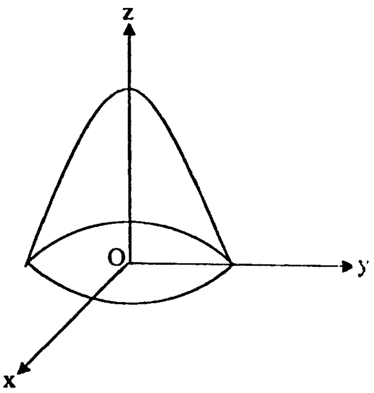
Rewriting the equation:

$$x^2 + y^2 - 2ax + az = 0$$

as

$$(x - a)^2 + y^2 = -a(z - a)$$

We see that the surface is a paraboloid with its vertex at (a , o , a) and axis parallel to z-axis and turned towards the negative direction of the same. It meets the plane z = 0 in the circle, C.



Given by:

$$x^2 + y^2 - 2ax = 0 \quad , \quad z = 0$$

(In the Fig, 0, is the point (a , 0 , 0), and ox , oy , oz are the lines parallel to the co-ordinate axes).

By Stoke's theorem, the given surface integral is equal to the line integral:

$$\int_C \underline{F} \cdot d\underline{r}$$

The circle G is given by:

$$x = a(1+\cos\theta) \quad , \quad y = a\sin\theta \quad , \quad z = 0$$

Along C:

$$\underline{F} = \left[a^2\sin^2\theta - a^2(1+\cos\theta)^2\right]\underline{i} + \left[a^2(1+\cos\theta)^2 - a^2\sin^2\theta\right]\underline{j}$$

$$+\left[a^2(1+\cos\theta)^2+a^2\sin^2\theta\right]\underline{k}$$

$$\therefore \underline{F}.\left[\frac{dx}{d\theta}\underline{i}+\frac{dy}{d\theta}\underline{j}+\frac{dz}{d\theta}\underline{k}\right]$$

$$=\left[a^2\sin^2\theta-a^2(1+\cos\theta)^2\right]\left[-a\sin\theta-a\cos\theta\right]$$

$$\therefore \int_C \underline{F}.d\underline{r}=\int_0^{2\pi}\underline{F}.\frac{dr}{d\theta}d\theta=2a^2\pi$$

Another method: By a application of Stoke's theorem, we see that the given integral:

$$=\int_{S_1}\text{curl}\,\underline{F}.\underline{n}\ dS$$

where S_1 is the plane region bounded by the circle C.

Here: $\underline{n}=\underline{k}$

Thus: $\text{curl}\,\underline{F}.\underline{n}=2(x-y)$

$$\therefore \text{the integral}=2\iint(x-y)dx\ dy$$

taken over S_1.

Changing to polar co-ordinates, so that:

$$x=a+r\cos\theta\quad,\quad y=r\sin\theta$$

we see that the integral:

$$= 2\int_0^a \int_0^{2\pi} (a + r\cos\theta - r\sin\theta) r \; d\theta \; dr$$

$$= 2a^2\pi$$

Ex.3: Apply Stoke's theorem to prove that:

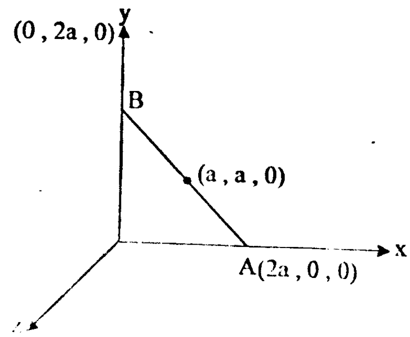
$$\int_C (y \; dx + z \; dy + x \; dz) = -2\sqrt{2} \, \pi \, a^2$$

where, C, is the curve given by:

$$x^2 + y^2 + z^2 - 2ax - 2ay = 0 \quad , \quad x + y = 2$$

and begins at the point (2a , 0 , 0) and goes at first be
plane.

Solution: It may be easily seen that the curve C is t
drawn on AB as diameter and lying in the plane x
which passes through A , B and is perpendicular to the



The given line integral:

$$= \int_C \left(y\underline{i} + z\underline{j} + x\underline{k}\right) = \int_S \text{curl } \left(y\underline{i} + z\underline{j} + x\underline{k}\right) . \underline{n} \ dS$$

where S is the surface of the circle referred to above. We have:

$$\text{curl } \left(y\underline{i} + z\underline{j} + x\underline{k}\right) = -\left(\underline{i} + \underline{j} + \underline{k}\right)$$

$$\underline{n} = \frac{1}{\sqrt{2}}\underline{i} + \frac{1}{\sqrt{2}}\underline{j}$$

The given line integral:

$$\int_S -\left(\underline{i} + \underline{j} + \underline{k}\right) . \left(\frac{1}{\sqrt{2}}\underline{i} + \frac{1}{\sqrt{2}}\underline{j}\right) dS$$

$$-\frac{2}{\sqrt{2}} \int_S dS = -\sqrt{2}\left(2a^2\pi\right) = -\sqrt{2}\,\pi\, a^2$$

Ex.4: Prove that:

$$\int_C \underline{r} \wedge d\underline{r} = 2\int_S d\underline{a}$$

where S is a diaphragm enclosing a circuit C.

We apply Stoke's theorem to the function $\underline{b} \wedge \underline{r}$ where $\underline{b}$ is any constant vector.

Thus:

$$\int_C \underline{b} \wedge \underline{r} . d\underline{r} = \int_S \text{curl}\left(\underline{b} \wedge \underline{r}\right) . d\underline{a}$$

Now:

$$\underline{\nabla} \wedge (\underline{b} \wedge \underline{r}) = \underline{b\nabla} . \underline{r} - \underline{b} . \underline{\nabla r} = 3\underline{b} - \underline{b} = 2\,\underline{b}$$

$$\int_C \underline{b} \wedge \underline{r} . d\underline{r} = 2\int_S \underline{b} . d\underline{a}$$

or $$\underline{b} . \int_C \underline{r} \wedge d\underline{r} = 2\underline{b} . \int_S d\underline{a}$$

or $$\underline{b} . \left[\int_C \underline{r} \wedge d\underline{r} - 2 \int_S d\underline{a} \right] = 0$$

$$\int_C \underline{r} \wedge d\underline{r} = 2\int_S d\underline{a}$$

for $\underline{b}$ is an arbitrary vector.

7- Green's Theorem:

If ϕ and ψ are two continuously different liable scalar point functions such that $\underline{\nabla}\,\phi$ and $\underline{\nabla}\,\psi$ are also continuously differentiable, then:

$$\int_V \left(\phi \nabla^2 \psi - \psi \nabla^2 \phi\right) dV = \int_S \left(\phi \underline{\nabla} \psi - \psi \underline{\nabla} \phi\right) . \underline{n}\, dS$$

where V is the region enclosed by a surface S.

Applying Gauss's Theorem to the function $\phi \nabla \psi$, we obtain:

$$\int_S \phi \nabla \psi \,.\, dS = \int_V \operatorname{div}(\phi \nabla \psi)\, dV$$

Also:

$$\begin{aligned}\operatorname{div}(\phi\nabla\psi) &= \underline{\nabla}.(\phi\underline{\nabla}\psi)\\ &= \underline{\nabla}\phi.\underline{\nabla}\psi + \phi\nabla^2\psi\end{aligned}$$

$$\int_S \phi\underline{\nabla}\psi.\underline{n}\, dS = \int_V \underline{\nabla}\phi.\underline{\nabla}\psi\, dV + \int_V \phi\nabla^2\psi\, dV \qquad (1)$$

or
$$\int_V \underline{\nabla}\phi.\underline{\nabla}\psi\, dV = \int_S \phi\underline{\nabla}\psi.\underline{n}\, dS - \int_V \phi\nabla^2\psi\, dV \qquad (2)$$

The result (2) is known as Green's first identity.

Interchanging ϕ and ψ, we get:

$$\int_S \psi\underline{\nabla}\phi.\underline{n}\, dS = \int_V \underline{\nabla}\psi.\underline{\nabla}\phi\, dV + \int_V \psi\nabla^2\phi\, dV \qquad (3)$$

Subtracting (3) from (1), we get:

$$\int_S (\phi\underline{\nabla}\psi - \psi\underline{\nabla}\phi).\underline{n}\, dS = \int_V (\phi\nabla^2\psi - \psi\nabla^2\phi)\, dV$$

Hence the theorem.

The result of the theorem can also be stated as:

$$\int_V (\phi\nabla^2\psi - \psi\nabla^2\phi)\, dV = \int_S \left(\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right) dS$$

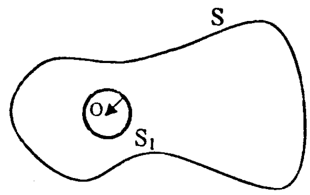
where $\partial\phi/\partial n$ and $\partial\psi/\partial n$ denote the directional derivatives of ϕ and ψ respectively along the outward drawn normal at any point of S.

Green's Formula:

Suppose that ϕ is a scalar point function which is twice continuously differentiable in a region V enclosed by a surface S.

Take any fixed point 0, within the region and let r denote the position vector of any point P relative to 0 and let:

$$OP = |r| = \underline{r}$$



We write:

$$\psi = \frac{1}{r}$$

so that, except at 0 , ψ is a twice continuously differentiable scalar function.

We surround 0 by a small sphere of radius ϵ_o Let S_1 denote the surface of this sphere and V_1 the region bounded by S and S_1. We have:

$$\underline{\nabla}\psi = -\frac{r}{r^3} \quad , \quad \nabla^2\psi = 0$$

We apply Green's theorem to the region V_1 enclosed by the surfaces S, S_1. Thus:

$$\int_V -\frac{1}{r}\nabla^2\phi\, dV = \int_S \phi\frac{\partial}{\partial n}\left(\frac{1}{r}\right) - \frac{1}{r}\frac{\partial \phi}{\partial n} dS$$

$$+\int_{S_1} \phi\frac{\partial}{\partial n}\left(\frac{1}{r}\right) - \frac{1}{r}\frac{\partial \phi}{\partial n} dS \qquad (1)$$

where the unit normal at any point of the sphere S_1 is directed towards 0.

At any point of S_1:

$$\frac{\partial}{\partial n}\left(\frac{1}{r}\right) = \underline{\nabla}\left(\frac{1}{r}\right) . \underline{n}$$

$$= \frac{r}{r^3} . -\frac{r}{r} = \frac{r^2}{r^4} = \frac{1}{r^2} = \frac{1}{\epsilon^2}$$

Now:

$$\int_{S_1} \phi\frac{\partial}{\partial n}\left(\frac{1}{r}\right) dS = \frac{1}{\epsilon^2}\int_{S_1} \phi \; dS$$

$$\lim_{\epsilon\to 0} \int_{S_1} \phi\frac{\partial}{\partial n}\left(\frac{1}{r}\right) dS = 4\pi\phi\,(0)$$

$$\lim_{\epsilon\to 0} \int_{S_1} \frac{1}{r}\frac{\partial \phi}{\partial n} dS = 0$$

Thus from (1), taking the limits when $\in \to 0$, we obtain:

$$\int_V \frac{1}{r}\nabla^2\phi\, dV = \int_S \left[\frac{\partial}{\partial n}\left(\frac{1}{r}\right) - \frac{1}{r}\frac{\partial \phi}{\partial n}\right] dS + 4\pi\phi(0)$$

or $$4\pi\phi(0) = \int_S \left[\frac{1}{r}\frac{\partial \phi}{\partial n} - \phi\frac{\partial}{\partial n}\left(\frac{1}{r}\right)\right] dS - \int_V \frac{1}{r}\nabla^2\phi\, dV$$

which may be re-written as:

$$4\pi\phi(0) = \int_S \left(\frac{1}{r}\underline{\nabla}\phi - \phi\underline{\nabla}\frac{1}{r}\right).d\underline{a} - \int_V \frac{1}{r}\nabla^2\phi\, dV$$

This result is known as Green's formula.

8- Harmonic Functions, Laplace's Equation and its solution:

Def. A function ϕ is said to be harmonic in a region, if it is twice continuously differentiable in the region and:

$$\nabla^2\phi = 0$$

at every point of the region.

Also the equation $\nabla^2\phi = 0$ is known as Laplace's equation.

Thus if ϕ De a harmonic function in a region V enclosed by a surface S, then, by Green's formula, we see that for any point O within the region:

$$4\pi\phi(0) = \int_S \left(\frac{1}{r}\underline{\nabla}\phi - \phi\underline{\nabla}\frac{1}{r}\right).d\underline{a} \qquad (1)$$

so that it is proved that the value of a harmonic function ϕ at any point within a region is known in terms of the values of ϕ and $\partial\phi/\partial n$ at any point of the surface enclosing the region.

9- Poisson's Equation and solution:

The equation:

$$\nabla^2\phi = -4\pi\rho$$

where ϕ is a scalar point function vanishing outside a finite region is known as Poisson's equation.

Consider a region V bounded by a surface S. Applying Green's formula to the region V; ϕ being a scalar point function satisfying (1), we have:

$$4\pi\phi(0) = \int_S \left(\frac{1}{r}\underline{\nabla}\phi - \phi\underline{\nabla}\frac{1}{r}\right).d\underline{a} + 4\pi\int_V \frac{\rho}{r}\,dV$$

We now let the region V tend to infinity in all directions so that the surface S also recedes to infinity.

We now assume that for sufficiently large values of r, ϕ is of the form k/r where k remains bounded. Then for sufficiently large values of r, $|\underline{\nabla}\phi|$ is of the form k/r^2.

$$\int_S \left(\frac{1}{r}\underline{\nabla}\phi - \phi\underline{\nabla}\frac{1}{r}\right).d\underline{a} \to 0$$

as y recedes to infinity in all directions.

Hence, we see that:

$$\phi(0) = \int_V \frac{\rho}{r} dV$$

where the volume integral is carried over the whole space. The volume integral carried over the whole space is the same as the volume integral carried over the region outside which ρ is zero.

Cor. Consider:

$$\nabla^2 \underline{F} = -4\pi \underline{f} \qquad (1)$$

Let: $\underline{f} = f_1 \underline{i} + f_2 \underline{j} + f_3 \underline{k}$

We write:

$$\underline{F} = F_1 \underline{i} + F_2 \underline{j} + F_3 \underline{k}$$

The equation (1) is equivalent to:

$$\nabla^2 F_1 = -4\pi f_1 \ , \ \nabla^2 F_2 = -4\pi f_2 \ , \ \nabla^2 F_3 = -4\pi f_3$$

Under suitable conditions imposed on f_1 , f_2 , f_3, we see that for any point O:

$$F_1(0) = \int \frac{f_1}{r} dV \quad , \quad F_2(0) = \int \frac{f_2}{r} dV \ ,$$

$$F_3(0) = \int \frac{f_3}{r} dV$$

These give:

$$F(0) = \int \frac{f}{r} dV$$

Problems

(1) Find the total work done in moving a particle in a force field given by $\underline{F} = xy\underline{i} - 5z\underline{j} + 10x\underline{k}$ along the curce $x = t^2 + 1$, $y = 2t^2$, $z = t^3$ from $t = 1$ to $t = 2$.

(2) If $\underline{F} = (2x + y)\underline{i} + (3y - x)\underline{j}$, evaluate $\int_C \underline{F} . dr$ along the curve C in the xy plane consisting of the straight lines from (0 , 0) to (2 , 0) and then to (3 ; 2).

(3) Prove that:

$$\underline{F} = (y^2 \cos x + z^3)\underline{i} + (2y \sin x - 4)\underline{j} + (3xz^2 + 2)\underline{k}$$

is a conservative force field and find the scalar potential for $\underline{F}$. Find also the work done in moving an object in this field from (0 , 1 , -1) to $(\pi/2, -1, 2)$.

(4) Find the work done in moving a particle in the force field $\underline{F} = 3x^2\underline{i} + (2xz - y)\underline{j} + z\underline{k}$ along:

(a) the straight line from (0 , 0 , 0) to (2 , 1 , 3). i

(b) the space curve $x = 2t$, $y = t$, $z = 4t^2$ from $t = 0$ to $t = 1$.

(5) If $\phi = 2xy^2z + x^2y$, evaluate $\int_C \phi \, dr$ where C is the curve $x = t$, $y = t^2$, $z = t^3$ from $t = 0$ to $t = 1$.

(6) Evaluate $\iint_S \underline{A}\,.\,\underline{n}\ dS$ over the entire surface s of the region bounded by the cylinder $x^2 + z^2 = 9n$, $x = 0$, $y = 0$, $z = 0$ and $y = 8$, $\underline{A} = 6z\underline{i} + (2x + y)\underline{j} - x\underline{k}$.

(7) If $\underline{F} = y\underline{i} + (x - 2xz)\underline{j} - xy\underline{k}$, evaluate $\iint_S (\underline{\nabla} \wedge \underline{F})\,.\,\underline{n}\ dS$ where s is the surface of the sphere $x^2 + y^2 + z^2 = a^2$ above the x y plane. Verify the result using stokes theorem.

(8) Evaluate:

$$\int_S (\underline{i}x + \underline{j}y + \underline{k}z)\,.\,d\underline{a}$$

where S denotes the surface of the cube bounded by the planes $x = 0$, $x = a$; $y = 0$, $y = a$; $z = 0$, $z = a$ by the application of Gauss's theorem and verify by direct manipulation.

(9) Evaluate:

$$\int_S (\underline{i}x + \underline{j}y + \underline{k}z)\,.\,d\underline{a}$$

where S is the part of the sphere $x^2 + y^2 + z^2 = 1$ above the xy plane.

(10) Show that:

$$\int_S (ax\underline{i} + by\underline{j} + cz\underline{k})\,.\,d\underline{a} = \frac{4}{3}\pi(a + b + c)$$

where S is the surface of the sphere $x^2 + y^2 + z^2 = 1$.

(11) Show that:

$$\int_S d\underline{a} = 0 \quad , \quad \int_S \underline{r} \, . \, d\underline{a} = 0$$

over any closed surface S.

(12) If $\underline{F}$ is normal to the surface S at each point, show that:

$$\int_V \text{curl}\, \underline{F} \;\; dV = 0$$

(13) Prove that:

$$\int_S \underline{n} \wedge (\underline{a} \wedge \underline{r}) \, dS = 2aV$$

where $\underline{a}$ is a constant vector and V is the volume enclosed by the closed surface S.

(14) Show that:

$$\int_S \underline{\nabla} \phi \wedge \underline{n} \; dS = 0$$

(15) Prove that:

$$\int_V \underline{F} \, . \, \underline{\nabla} \phi = \int_S \phi \underline{F} \, . \, d\underline{a} - \int_V \phi \underline{\nabla} \, . \, \underline{F} \, dV$$

(16) Prove that:

$$\int_V . \underline{\nabla} \phi \, . \, \text{curl}\, \underline{F} \, dV = \int_S \left(\underline{F} \wedge \underline{\nabla} \phi \right) . \, d\underline{a}$$

Solved Examples

(1) Given Maxwell's equations in empty space:

$$\nabla . \underline{E} = 0 \quad , \quad \nabla . \underline{H} = 0$$

$$\nabla \wedge \underline{E} = -\frac{\partial \bar{H}}{\partial t} \quad , \quad \nabla \wedge \underline{H} = -\frac{\partial \bar{E}}{\partial t}$$

Solution:

Show that $\underline{E}$ and $\underline{H}$ satisfy.

$$\nabla^2 \underline{u} = \frac{\partial^2 \bar{H}}{\partial t^2}$$

$$\nabla \wedge (\nabla \wedge \underline{E}) = \frac{\partial}{\partial t}(\nabla \wedge \underline{H}) = -\frac{\partial}{\partial t}\left(\frac{\partial \underline{E}}{\partial t}\right) = -\frac{\partial^2 \underline{E}}{\partial t^2}$$

i.e. $\quad \nabla(\nabla . \underline{E}) - \nabla^2 \underline{E} = -\dfrac{\partial^2 \underline{E}}{\partial t^2}$

but $\quad \nabla . \underline{E} = 0$ then $\nabla^2 \underline{E} = \dfrac{\partial^2 \underline{E}}{\partial t^2} \qquad (1)$

Similarly $\nabla \wedge (\nabla \wedge \underline{H}) - \nabla^2 \underline{H} = \dfrac{\partial}{\partial t}\left(-\dfrac{\partial \underline{H}}{\partial t}\right) = -\dfrac{\partial^2 \underline{H}}{\partial t^2}$

Then $\quad \nabla^2 \underline{H} = \dfrac{\partial^2 \underline{H}}{\partial t^2}$ since $\nabla . \underline{H} = 0$

Hence $\underline{E}$ and $\underline{H}$ satisfy the wave equation:

$$\frac{\partial^2 \underline{u}}{\partial x^2} + \frac{\partial^2 \underline{u}}{\partial y^2} + \frac{\partial^2 \underline{u}}{\partial z^2} = \frac{\partial^2 \underline{u}}{\partial t^2}$$

(2) Show that if ϕ is a solid spherical harmonic of degree n, then $\dfrac{\phi}{r^{2n+1}}$ is a solid spherical harmonic of degree $-(n+1)$.

Solution:

Since
$$\begin{aligned}\nabla^2(\phi\psi) &= \nabla . \nabla(\phi\psi)\\ &= \nabla . [\phi\nabla\psi + \psi\nabla\phi]\\ &= \nabla . (\phi\nabla\psi) + \nabla . (\psi\nabla\phi)\\ &= \phi\nabla . \nabla\psi + \nabla\phi . \nabla\psi\\ &\quad + \psi\nabla . \nabla\phi + \nabla\psi . \nabla\phi\\ &= \phi\nabla^2\psi + 2\nabla\phi . \nabla\psi + \psi\nabla^2\phi\end{aligned}$$

put $\quad \psi = \dfrac{1}{r^{2n+1}}$ then

$$\nabla^2(\phi\psi) = \nabla^2 \frac{\phi}{r^{2n+1}} = \phi \nabla^2 r^{-(2n+1)} + \frac{1}{r^{2n+1}} \nabla^2 \phi$$

$$+ 2\nabla r^{-(2n+1)} . \nabla \phi \qquad (1)$$

Since ϕ is harmonic $\nabla^2\phi = 0$ (2)

ϕ of degree $n \rightarrow \underline{r} . \nabla \phi = n\phi$ (3)

$$\nabla r^{-(2n+1)} = \frac{\bar{r}}{r} \frac{d}{dr} r^{-(2r+1)}$$

$$= -(2n+1) r^{-(2n+3)} \underline{r} \qquad (4)$$

$$\nabla^2 r^{-(2n+1)} = -(2n+1)\left[\nabla . \underline{r}\, r^{-(2n+3)}\right]$$

$$= -(2n+1)\left(r^{-(2n+3)} \nabla . \underline{r} + \nabla r^{-(2n+3)} . \underline{r}\right)$$

$$= -(2n+1)\left(3r^{-(2n+3)} - (2n+3) r^{-(2n+5)} \underline{r} . \underline{r}\right)$$

$$= -(2n+1)\left(\frac{3}{r^{2n+3}} - \frac{(2n+3)}{r^{2n+3}}\right)$$

$$= +2n(2n+1) / r^{2n+3} \qquad (5)$$

from (1), (5):

$$\nabla^2\left(\frac{\phi}{r^{2n+1}}\right) = \frac{2n(2n+1)\phi}{r^{2n+3}} - \frac{2(2n+1)}{r^{2r+3}} \underline{r} . \nabla \phi$$

i.e. $$\nabla^2\left(\frac{\phi}{r^{2n+1}}\right)=\frac{2(2n+1)}{r^{2n+3}}\left(n\phi-\underline{r}.\nabla\phi\right)$$

$$=\frac{2(2n+1)}{r^{2n+3}}\left(n\phi-n\phi\right)=0$$

i.e. $\frac{\phi}{r^{2n+1}}$ is a solid spherical harmonic.

For example: if $\phi=\frac{5\cos^3\theta-3\cos\theta}{r^4}$

Then $\phi=\frac{5r^3\cos^3\theta-3r^3\cos\theta}{r^7}=\frac{f}{r^7}$

Also satisfy laplace equ.

$$f=5r^3\cos^3\theta-3r^3\cos\theta=5z^3-3r^2\left(r\cos\theta\right)$$

$$=5z^3-3z\left(x^2+y^2+z^2\right)=2z^3-3z\left(x^2+y^2\right)$$

i.e. $$\nabla^2 f=-6z-6z+12z=0$$

Since f is homogenous equation of degree 3 and $\nabla^2 f=0$, then $\frac{f}{r^{2x3+1}}=\frac{f}{r^7}$ also satisfy laplace equation.

(3) Prove that:

a- $\nabla . \underline{r}\, f(r) = r\dfrac{df}{dr} + 3f$

b- $\nabla . (\underline{a} \wedge \underline{r}) = 0$

c- $\nabla \wedge \underline{r}\, f(r) = 0$

Solution:

a- $\nabla . (\phi \underline{A}) = \phi \nabla . \underline{A} + \nabla \phi . \underline{A}$

$$\nabla . (\underline{r}\, f(r)) = f(r) \nabla . \underline{r} + \nabla f(r) . \underline{r}$$

but $\nabla . \underline{r} = 3$, $\nabla f(r) = \underline{n}\dfrac{df}{dr} = \dfrac{\underline{r}}{r}\dfrac{df}{dr}$

$$\nabla . (\underline{r}\, f(r)) = 3f + \frac{\underline{r}}{r}\frac{df}{dr} . \underline{r}$$

$$= 3f + r\frac{df}{dr}$$

b- Since $\underline{a} \wedge \underline{r} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ a_1 & a_2 & a_3 \\ x & y & z \end{vmatrix}$

$$= (a_2 y - a_3 z)\underline{i} + (a_3 x - a_1 z)\underline{j} + (a_1 y - a_2 x)\underline{k}$$

$$\nabla . (\underline{a} \wedge \underline{r}) = \frac{\partial}{\partial x}(a_2 y - a_3 z) + \frac{\partial}{\partial y}(a_3 x - a_1 z)$$

$$+\frac{\partial}{\partial z}(a_1 y - a_2 x)$$

$$= 0 + 0 + 0 = 0$$

c- Since $\nabla \wedge (\underline{r}\ f(r)) = f(r)\nabla \wedge \underline{r} + \nabla f(r) \wedge \underline{r}$

but $\quad \nabla \wedge \underline{r} = \underline{o} \quad , \quad \nabla f(r) = \frac{\underline{r}}{r}\frac{df}{dr}$

$$\nabla \wedge \underline{r}\, f(r) = \underline{o} + \frac{\underline{r}}{r}\frac{df}{dr} \wedge \underline{r}$$

$$= \frac{1}{r}\frac{df}{dr}(\underline{r} \wedge \underline{r}) = \underline{o}$$

Since $\quad \underline{r} \wedge \underline{r} = \underline{o}$

(4) Find a , b , c such that:

$$\underline{V} = (x + 2y + az)\underline{i} + (bx - 3y - z)\underline{j} + (4x + cy + 2z)\underline{k}$$

irrotational and find its scalar potential show that $\underline{V}$ is solenoidal.

Solution: Since $\underline{V}$ is irrotational $\nabla \wedge \underline{V} = \underline{o}$

i.e. $\quad \underline{a} \wedge \underline{r} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix} = \underline{o}$

$$\begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x+2y+az & bx-3y-z & 4x+cy+2z \end{vmatrix} = \underline{o}$$

$$c = -1 \quad , \quad a = 4 \quad , \quad b = 2$$

i.e. $\underline{V} = (x+2y+4z)\underline{i} + (2x-3y-z)\underline{j} + (4x-y+2z)\underline{k}$

$\because \nabla \wedge \underline{V} = \underline{o}$ there exist scalar point function ϕ such that $\underline{V} = \nabla\phi$.

i.e. $$\frac{\partial\phi}{\partial x} = x+2y+4z \Rightarrow \phi = \frac{x^2}{2} + 2xy + 4xz + f_1(x,z)$$

$$\frac{\partial\phi}{\partial y} = 2x-3y-z \Rightarrow \phi = 2xy - \frac{3}{2}y^2 - zy + f_2(x,z)$$

$$\frac{\partial\phi}{\partial z} = 4x-y+2z \Rightarrow \phi = 4xy - yz + z^2 + f_3(x,y)$$

Hence $\phi = \dfrac{x^2}{2} - \dfrac{3}{2}y^2 + z^2 + 2xy + 4xz - yz + c$

The vector is solenoidal if div $\underline{V} = 0$

$$\text{div}\,\underline{V} = \frac{\partial}{\partial x}V_x + \frac{\partial}{\partial y}V_y + \frac{\partial}{\partial z}V_z$$

$$= \frac{\partial}{\partial x}(x+2y+4z)+\frac{\partial}{\partial y}(2x-3y-z)+\frac{\partial}{\partial z}(4x-y+2z)$$

$$= 1-3+2=0$$

(5) Show that if $\phi(x,y,z)$ is solution of Laplace's equation, then $\nabla\phi$ is a vector which is both solenoidal and irrotational. Since ϕ is solution of Laplace equ.

i.e. $\nabla^2\phi = 0 \Rightarrow \nabla.\nabla\phi = 0$

i.e. $\nabla\phi$ is solenoidal vector. Also since:

$$\nabla \wedge \nabla\phi = \text{curl grad}\,\phi = \underline{o}$$

therefore $\nabla\phi$ is irrotational.

(6) If $\underline{A} = f(r)\underline{k} \wedge \underline{r}$, $\underline{r} = x\underline{i} + y\underline{j}$, find f® if the field is:

(a) irrotational (b) solenoidal

Solution:

(a) If the field is irrotational, then:

$$\text{curl}\,\underline{A} = \nabla \wedge \underline{A} = \underline{o}$$

i.e. $\nabla \wedge f(r)\underline{k} \wedge \underline{r} = 0$

$$\underline{k} \wedge \underline{r} = \underline{k} \wedge (x\underline{i} + y\underline{j}) = x\underline{j} - y\underline{i}$$

$$\nabla \wedge f(r)(\underline{k} \wedge \underline{r}) = \nabla f(r) \wedge (\underline{k} \wedge \underline{r})$$

$$+ f(r)\nabla \wedge (\underline{k} \wedge \underline{r}) = 0 \qquad (1)$$

$$\nabla f(r) = \frac{\underline{r}}{r}\frac{df}{dr} \quad , \quad \nabla \wedge (\underline{k} \wedge \underline{r}) = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & x & 0 \end{vmatrix}$$

$$= 2\underline{k}$$

$$(1) \Rightarrow \left(\frac{1}{r}\frac{df}{dr}\right)\underline{r} \wedge (\underline{k} \wedge \underline{r}) + 2f(r)\underline{k} = \underline{o}$$

$$\frac{1}{r}\frac{df}{dr}\left(r^2\underline{k} - (\underline{r}.\underline{k})\underline{r}\right) + 2f(r)\underline{k} = 0$$

but $\quad \underline{r}.\underline{k} = (x\underline{i} + y\underline{j}).\underline{k} = 0$

$$\therefore \left(r\frac{df}{dr} + 2f\right)\underline{k} = \underline{o} \qquad r\frac{df}{dr} + 2f = 0$$

i.e. $\quad \dfrac{df}{f} = -\dfrac{2dr}{r}$

$$\ln f = -2\ln r + \ln c = \ln\left(c/r^2\right)$$

$f = \dfrac{c}{r^2}$ where c is arbitrary constant, $r^2 = x^2 + y^2$.

(b) If the field is solenoidal:

i.e. $\nabla . \underline{A} = 0$

$$\nabla . \left(f(\underline{r})\underline{k} \wedge \underline{r} \right) = 0$$

i.e. $f(r)\nabla . \left(\underline{k} \wedge \underline{r}\right) + \nabla f(\underline{r}) . \underline{k} \wedge \underline{r} = 0$

but $\nabla . \left(\underline{k} \wedge \underline{r}\right) = \underline{r} . \nabla \wedge \underline{k} - \underline{k} . \nabla \wedge \underline{r}$

$$= \underline{r} . \nabla \wedge \underline{k}$$

Since curl $\underline{r} = 0$

$$\nabla \wedge \underline{k} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & 1 \end{vmatrix} = \underline{o}$$

$$\nabla . \left(\underline{k} \wedge \underline{r}\right) = 0$$

$$(2) \Rightarrow \frac{r}{r}\frac{df}{dr} . \underline{k} \wedge \underline{r} = 0$$

$$\frac{df}{dr}\left(\underline{r} . \underline{k} \wedge \underline{r}\right) = 0$$

but $\underline{r} . \underline{k} \wedge \underline{r} = 0 \Rightarrow \frac{df}{dr} \neq 0$

If we take $\dfrac{df}{dr} = \text{const } c$

$$df = c\, dr \Rightarrow f = cr$$

(7) Consider the circular helix, defined parametrically as:

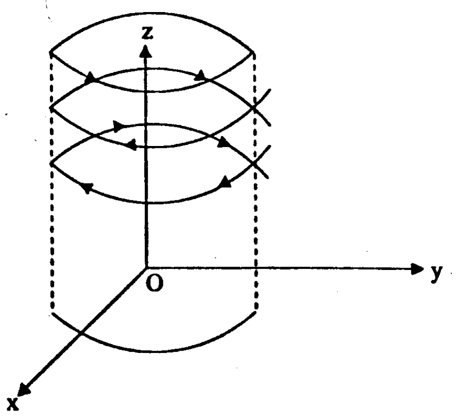
$$\underline{r} = (a\cos t, a\sin t, bt) \qquad (1)$$

where a , b are constants. Equating the components,

$$x = a\cos t \quad , \quad y = a\sin t \quad , \quad z = b\,t$$

and so for all t:

$$x^2 + y^2 = a^2$$



The curve therefore lies on the surface of a circular cylinder of radius a and axis oz. It spirals around the z-axis, as shown in Figure. For this curve.

$$\frac{ds}{dt} = \left(\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 \right)^{1/2}$$

$$= \left(a^2 + b^2\right)^{1/2}$$

Hence $\quad \bar{T} = \frac{d\bar{r}}{ds} = \frac{d\bar{r}}{dt}\frac{dt}{ds}$

$$= \frac{1}{\left(a^2 + b^2\right)^{1/2}}(-a\sin t, a\cos t, b) \qquad (2)$$

Also $\quad k\bar{N} = \frac{d\bar{T}}{ds} - \frac{a}{\left(a^2 + b^2\right)}(\cos t, \sin t, 0) \qquad (3)$

It follows that the principal unit normal $\bar{N}$ is always parallel to the xy-plane.

Also, equating the magnitudes of the two sides of the curvature is:

$$k = |a| / \left(a^2 + b^2\right) \qquad (4)$$

From equations (2), (3):

$$\bar{B} = \bar{T} \times \bar{N} = \frac{a}{k\left(a^2 + b^2\right)^{3/2}}(b\sin t, -b\cos t, a)$$

Thus $\quad \tau\overline{N} = -\frac{d\overline{B}}{ds} = -\frac{ab}{k(a^2+b^2)^2}(\cos t, \sin t, 0)$

And using (3) this reduces to:

$$\tau\overline{N} = \frac{b}{a^2+b^2}\overline{N}$$

The torsion of the helix is therefore:

$$\tau = b/a^2 + b^2 \qquad (5)$$

Notice the when $b = 0$, the curve reduces to a circle of radius a in the xy-plane. As expected, the curvature is then $1/a$ (from (4)), and the torsion is zero (from (5)).

(8) A space curve is give by:

$$x = t \quad , \quad y = t^2 \quad , \quad z = t^3$$

Find the curvature k and the torsion.

The Position vector is:

$$\underline{r}(t) = (t, t^2, t^3) \qquad (1)$$

Then $\quad \dot{\overline{r}} = \frac{d\overline{r}}{dt} = (1, 2t, 3t^2) \qquad (2)$

From eqn (2), find $\dot{s}$.

$$\dot{s} = \left|\dot{\vec{r}}\right| = \sqrt{1+4t^2+9t^4}$$

The unit tangent vector is:

$$\vec{T} = \frac{d\vec{r}}{ds} = \frac{\dot{\vec{r}}}{\dot{s}}$$

Thus:

$$\vec{T} = \left(\frac{1}{\sqrt{1+4t^2+9t^4}}, \frac{2t}{\sqrt{1+4t^2+9t^4}}, \frac{3t^2}{\sqrt{1+4t^2+9t^4}}\right) \qquad (3)$$

Since $k = \left|\frac{d\vec{T}}{ds}\right|$

And $\vec{T} = \vec{T}(t)$, compute:

$$\frac{d\vec{T}}{ds} = \frac{\dot{\vec{T}}}{\dot{s}}$$

Differentiating eqn (3):

$$\dot{\vec{T}} = \left(\frac{-4t-18t^2}{\left(1+4t^2+9t^4\right)^{3/2}}, \frac{2-18t^4}{\left(1+4t^2+9t^4\right)^{3/2}}, \frac{6t+12t^3}{\left(1+4t^2+9t^4\right)^{3/2}}\right)$$

Thus $\frac{\dot{\vec{T}}}{\dot{s}}$ can be written.

$$\frac{\vec{T}}{\dot{s}} = \frac{d\vec{T}}{ds} =$$

$$= \left(\frac{-4t - 18t^2}{\left(1 + 4t^2 + 9t^4\right)^2}, \frac{2 - 18t^4}{\left(1 + 4t^2 + 9t^4\right)^2}, \frac{6t + 12t^3}{\left(1 + 4t^2 + 9t^4\right)^2} \right)$$

then $\quad k = \left| \frac{d\vec{T}}{ds} \right| = \frac{2\sqrt{1 + 9t^2 + 9t^4}}{\left(1 + 4t^2 + 9t^4\right)^{3/2}} \qquad (4)$

The normal vector $\vec{N}$ is given by:

$$\vec{N} = \frac{1}{k} \frac{d\vec{T}}{ds} \qquad (5)$$

Substituting eqn (4) and a bout $\frac{\vec{T}}{\dot{s}}$ into equation (5) ginen:

$$\vec{N} = \frac{1}{\sqrt{1 + 9t^2 + 9t^4}\sqrt{1 + 4t^2 + 9t^4}} \left(-2t - 9t^3, 1 - 9t^4, 3t + 6t^3\right)$$

From the Frenet formula:

$$\frac{d\vec{B}}{ds} = -\tau \vec{N} \Rightarrow \tau = \left| \frac{d\vec{B}}{ds} \right|$$

The binormal vector $\vec{B}$ is given by:

$$\vec{B} = \vec{T} \times \vec{N}$$

Substituting a bout $\bar{T}, \bar{N}$, we get:

$$\bar{B} = \frac{1}{\sqrt{1+4t^2+9t^4}}\left(1,2t,3t^2\right)\left(\frac{-2t-9t^3, 1-9t^4, 3t+6t^3}{\sqrt{1+9t^2+9t^4}\sqrt{1+4t^2+9t^4}}\right)$$

Then $\quad \tau = \left|\frac{d\bar{B}}{ds}\right| = \frac{\dot{\bar{B}}}{\dot{s}}$

Differentiating $\bar{B}$ and substituting $\dot{\bar{B}}$ we obtain.

$$\tau = \frac{3}{\left(1+9t^2+9t^4\right)}$$

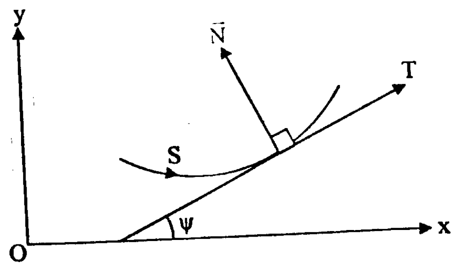
(9) Show that for a plane curve in the xy-plane, if the unit tangent in the direction of s increasing make an angle ψ with the positive x-axis then the radius of curvature ρ is given by:

$$\rho = \left|ds/d\psi\right|$$

In terms of the angle ψ:

$$\bar{T} = \left(\cos\psi, \sin\psi, 0\right)$$

and so $\quad \frac{d\bar{T}}{ds} = \frac{d\bar{T}}{d\psi}\frac{d\psi}{ds} = \frac{d\psi}{ds}\left(-\sin\psi, \cos\psi, 0\right)$



It follows by comparison with the formula:

$$\frac{d\vec{T}}{ds} = k\vec{N} \text{ that } k = \left|\frac{d\psi}{ds}\right|$$

Thus the radius of curvature is:

$$\rho = \frac{1}{k} = \left|\frac{ds}{d\psi}\right|$$

Note that:

$$N = \begin{cases} (-\sin\psi, \cos\psi, 0) & \text{when } \dfrac{ds}{d\psi} > 0 \\ (\sin\psi, -\cos\psi, 0) & \text{when } \dfrac{ds}{d\psi} < 0 \end{cases}$$

(10) Acurve in space is given by $\vec{r} = \vec{r}(t)$, show that if $\left|\dot{\vec{r}}\right| \neq 0$ then the curvature k is equal to:

$$k = \frac{\left|\dot{\bar{r}} \times \ddot{\bar{r}}\right|}{\left|\dot{\bar{r}}\right|^3}$$

$$\dot{\bar{r}} = \frac{d\bar{r}}{dt} = \frac{d\bar{r}}{ds}\frac{ds}{dt} = \frac{ds}{dt}\bar{T}$$

where $\bar{T} = \dfrac{d\bar{r}}{ds}$

also $\ddot{\bar{r}} = \dfrac{d^2}{dt^2}\bar{r} = \dfrac{ds}{dt}\dot{\bar{T}} + \dfrac{d^2s}{dt^2}\bar{T}$

Thus $\dot{\bar{r}} \times \ddot{\bar{r}} = \dfrac{ds}{dt}\bar{T} \times \left(\dfrac{ds}{dt}\dot{\bar{T}} + \dfrac{d^2s}{dt^2}\bar{T}\right)$

$$= \left(\frac{ds}{dt}\right)^2 \left(\bar{T} \wedge \dot{\bar{T}}\right)$$

but $\dot{\bar{T}} = \dfrac{d\bar{T}}{dt} = \dfrac{d\bar{T}}{ds}\dfrac{ds}{dt} = k\bar{N}\dfrac{ds}{dt}$

Since $\dfrac{d\bar{T}}{ds} = k\bar{N}$

Hence $\dot{\bar{r}} \times \ddot{\bar{r}} = \left(\dfrac{ds}{dt}\right)^3 k\left(\bar{T} \wedge \bar{N}\right)$

The binormal $\bar{B}$ is defined as:

$$\vec{B} = \vec{T} \wedge \vec{N} \quad , \quad |\vec{B}| = 1$$

Therefore $$\dot{\vec{r}} \times \ddot{\vec{r}} = \left(\frac{ds}{dt}\right)^3 k\,\vec{B}$$

$$|\dot{\vec{r}} \times \ddot{\vec{r}}| = k\left|\left(\frac{ds}{dt}\right)^3\right|$$

Note $$\frac{ds}{dt} = |\dot{\vec{r}}| \Rightarrow k\frac{|\dot{\vec{r}} \times \ddot{\vec{r}}|}{|\dot{r}|^3}$$

(11) Find the potential of a vector field specified in cylindrical coordinates:

$$\vec{A} = \left(\frac{1}{\rho}\tan^{-1} z + \cos\phi\right)\vec{e}_\rho - \sin\phi\,\vec{e}_\phi + \frac{\ln\rho}{1+z^2}\vec{e}_z$$

First must be equal to zero:

$$\nabla \wedge \vec{A} = \frac{1}{\rho}\begin{vmatrix} \vec{e}_\rho & \rho\vec{e}_\phi & \vec{e}_z \\ \dfrac{\partial}{\partial\rho} & \dfrac{\partial}{\partial\phi} & \dfrac{\partial}{\partial z} \\ \dfrac{1}{\rho}\tan^{-1} z + \cos\phi & -\rho\sin\phi & \dfrac{\ln\rho}{1+z^2} \end{vmatrix}$$

i.e. the field is a potential one.

The potential $V = V(\rho,\phi,z)$ is a solution of t[illegible]

system of differential equation with partial deriv[illegible]

$$\vec{A} = \nabla V$$

$$\frac{\partial V}{\partial \rho} = \frac{1}{\rho}\tan^{-1} z + \cos\phi$$

$$\frac{\partial V}{\partial \phi} = -\rho \sin\phi$$

$$\frac{\partial V}{\partial z} = \frac{\ln\rho}{1+z^2}$$

Integrating with respect to ρ, we find from the first of th[illegible] equations:

$$V = \ln\rho \tan^{-1} z + \rho\cos\phi + C(\phi,z)$$

(12) Prove that vector E whose components in terms of t[illegible] cylindrical cords are:

$$E_r = \left(1 - \frac{a^2}{\rho^2}\right)\cos\theta$$

$$E_\theta = -\left(1 + \frac{a^2}{\rho^2}\right)\sin\theta$$

where a is const. is irrota[illegible]al [illegible]d find its scal[illegible] pot[illegible] solenodial.

Solution:

(i) $\underline{\nabla} \wedge \underline{E} = 0$

$$\underline{\nabla} \wedge \underline{E} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 e_1 & h_2 e_2 & h_3 e_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1 E_1 & h_2 E_2 & h_3 E_3 \end{vmatrix}$$

$u_1 = \rho \qquad h_1 = 1$

$u_2 = \phi \qquad h_2 = \rho$

$u_3 = z \qquad h_3 = 1$

$$\underline{\nabla} \wedge \underline{E} = \frac{1}{\rho} \begin{vmatrix} \underline{e}_\rho & r\underline{e}_\phi & \underline{e}_z \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ \left(1 - \frac{\partial^2}{\rho^2}\right)\cos\phi & -\rho\left(1 + \frac{\partial^2}{\rho^2}\right)\sin\phi & 0 \end{vmatrix}$$

$$= \frac{1}{\rho}\left[\frac{\partial}{\partial \rho}\left(-r\left(1 + \frac{\partial^2}{\rho^2}\right)\sin\theta\right) - \frac{\partial}{\partial \theta}\left(\left(1 - \frac{\partial^2}{\rho^2}\right)\cos\theta\right)\right]\underline{e}_z$$

$$\underline{\nabla} \wedge \underline{E} = \frac{1}{\rho}\left[\left(-1 + \frac{\partial^2}{\rho^2}\right)\sin\phi + \left(1 - \frac{\partial^2}{\rho^2}\right)\sin\phi\right]\underline{e}_z$$

$$=\frac{1}{r}\left[-\left(-1+\frac{\partial^2}{\rho^2}\right)\sin\phi+\left(1-\frac{\partial^2}{\rho^2}\right)\sin\phi\right]\underline{e}_z$$

$$\underline{\nabla}\wedge\underline{E}=0$$

$\therefore$ $\underline{E}$ is irrotational.

(ii) $\phi=??$

$$d\phi=\nabla\phi\,.\,d\underline{r}$$

Let: $$E=\nabla\phi$$

$$d\phi=E\,.\,d\underline{r}$$

$$\underline{E}=E_r\underline{e}_r+E_\theta\underline{e}_\theta+E_z\underline{e}_z$$

$$d\underline{r}=dr\underline{e}_r+r\,d\theta\,\underline{e}_\theta+dz\,\underline{e}_z$$

$$\underline{E}\,.\,d\underline{r}=E_r dr+r\,E_\theta d\theta+E_z dz \qquad (1)$$

$$\phi=\int\left(1-\frac{\partial^2}{\rho^2}\right)\cos\phi\,dr-\int\rho\left(1+\frac{\partial^2}{\rho^2}\right)\sin\phi\,d\theta+C$$

$$=\cos\theta\int\left(1-\frac{\partial^2}{\rho^2}\right)d\rho-\left(\rho+\frac{\partial^2}{\rho}\right)\sin\phi\,d\phi+C$$

$$\phi=\cos\theta\left(\rho+\frac{\partial^2}{\rho}\right)+\left(\rho+\frac{\partial^2}{\rho}\right)\cos\phi+C$$

$$\phi=2\left(\rho+\frac{\partial^2}{\rho}\right)\cos\phi+C$$

(iii) $\underline{\nabla}.\underline{E}=0$

$$\underline{\nabla}.\underline{E}=\frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}(E_1h_2h_3)+\frac{\partial}{\partial u_2}(h_1E_2h_3)+\frac{\partial}{\partial u_3}(h_1h_2E_3)\right]$$

$$\underline{\nabla}.\underline{E}=\frac{1}{\rho}\left[\frac{\partial}{\partial\rho}\rho\left(1-\frac{\partial^2}{\rho^2}\right)\cos\phi+\frac{\partial}{\partial\phi}\left(-\left(1+\frac{\partial^2}{\rho^2}\right)\sin\phi\right)\right]$$

$$=\frac{1}{\rho}\left[\frac{\partial}{\partial\rho}\left(\rho-\frac{\partial^2}{\rho}\right)\cos\phi-\frac{\partial}{\partial\phi}\left(1+\frac{\partial^2}{\rho^2}\right)\sin\phi\right]$$

$$=\frac{1}{r}\left[\left(1+\frac{\partial^2}{\rho^2}\right)\cos\phi-\left(1+\frac{\partial^2}{\rho^2}\right)\cos\phi\right]=0$$

$\underline{\nabla}.\underline{E}=0$

$\therefore$ $\underline{E}$ **is solenodial.**

(13) Given that $f(r,\theta,\phi)=ar^2\sin\theta\cos^2\phi$ where (r,θ,ϕ) are spherical cords. Calculate the derivative of f in the direction $\underline{S}=\frac{1}{4}\left[2\underline{e}_r+\sqrt{3}\underline{e}_\theta+2\underline{e}_\phi\right]$ at the point $\left(a,\frac{\pi}{3},\frac{\pi}{4}\right)$.

Solution:

$$d.d=\nabla F.(\underline{S})$$

$$\nabla F=\frac{e_1}{h_1}\frac{\partial f_1}{\partial u_1}+\frac{e_2}{h_2}\frac{\partial f_2}{\partial u_2}+\frac{e_3}{h_3}\frac{\partial f_3}{\partial u_3}$$

$$u_1=r \qquad h_1=1$$

$$u_2 = \theta \qquad h_2 = r$$

$$u_3 = \phi \qquad h_3 = r\sin\theta$$

$$\nabla F = \frac{\partial}{\partial r}\left(ar^2 \sin\theta \cos^2\phi\right)\underline{e}_r + \frac{1}{r}\cdot\frac{\partial}{\partial\theta}\left(ar^2\sin\theta\cos^2\phi\right)\underline{e}_\theta$$

$$+\frac{1}{r\sin\theta}\cdot\frac{\partial}{\partial\phi}\left(ar^2\sin\theta\cos^2\phi\right)\underline{e}_\phi$$

$$\nabla F = 2ar^2\sin\theta\cos^2\phi\,\underline{e}_r + ar\cos\theta\cos^2\phi\,\underline{e}_\theta - 2ar\sin\phi\cos\phi\underline{e}_\phi$$

$$\text{at}\left(a,\frac{\pi}{3},\frac{\pi}{4}\right)$$

$$\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2} \quad , \quad \cos\frac{\pi}{3} = \frac{1}{2} \quad , \quad \sin\frac{\pi}{4} = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

$$\nabla F = 2a^2\,.\frac{\sqrt{3}}{2}\cdot\frac{1}{2}\underline{e}_r + a^2\,.\frac{1}{2}\cdot\frac{1}{2}\underline{e}_\theta - 2a^2\,.\frac{1}{2}\underline{e}_\phi$$

$$\nabla F = \frac{\sqrt{3}}{2}a^2\,\underline{e}_r + \frac{1}{4}a^2\underline{e}_\theta - a^2\underline{e}_\phi$$

$$d\,.\,d = \nabla F\,.\,(\underline{S})$$

$$d\,.\,d = \left(\frac{\sqrt{3}}{2}a^2\,\underline{e}_r + \frac{1}{4}a^2\underline{e}_\theta - a^2\underline{e}_\phi\right)\cdot\left(\frac{1}{4}\left(3\,\underline{e}_r + \sqrt{3}a^2\underline{e}_\theta + 2\underline{e}_\phi\right)\right)$$

$$d\,.\,d = \frac{1}{4}\left(\frac{3\sqrt{3}}{2}a^2 + \frac{\sqrt{3}}{4}a^2 - 2a^2\right)$$

$$= \frac{1}{4}\left(\frac{7\sqrt{3}}{4}a^2 - 2a^2\right)$$

$$d\,.\,d = \left(\frac{7\sqrt{3}}{16} - \frac{1}{2}\right)a^2$$